Calculus for Economics, Commerce and Management

Assignment 6B - Subjective (Solutions)

(1) [4 marks]
Find the local maxima/minima and global maxima/minima, if any of the following functions. (Full marks awarded for finding just the local maxima/minima.)

(a) \( f(x) = \frac{3x}{x^2 + 1} \)

Solution.

(1) (Using Second Derivative Test) We have

\[
f'(x) = \frac{(x^2 + 1)(3) - (3x)(2x)}{(x^2 + 1)^2} = \frac{-3x^2 + 3}{(x^2 + 1)^2}
\]

and

\[
f''(x) = \frac{(x^2 + 1)^2(-6x) - (-3x^2 + 3)(4x)(x^2 + 1)}{(x^2 + 1)^4} = \frac{(x^2 + 1)(-6x) - (-3x^2 + 3)(4x)}{(x^2 + 1)^3} = \frac{6x^3 - 18x}{(x^2 + 1)^3}.
\]

So

\[
f'(x) = 0 \iff \frac{-3x^2 + 3}{(x^2 + 1)^2} = 0 \implies x^2 = 1 \implies x = \pm 1.
\]

Further \( f''(-1) = \frac{12}{8} = \frac{3}{2} > 0 \) and \( f''(1) = -\frac{12}{8} = -\frac{3}{2} < 0 \). So by second derivative test, \( f \) has a local minimum at -1 and a local maximum at 1. Note that \( f(-1) = \frac{-3}{2} \) and \( f(1) = \frac{3}{2} \).

(2) (Using First Derivative Test) We have

\[
f'(x) = \frac{(x^2 + 1)(3) - (3x)(2x)}{(x^2 + 1)^2} = \frac{-3x^2 + 3}{(x^2 + 1)^2}
\]

So

\[
f'(x) = 0 \iff \frac{-3x^2 + 3}{(x^2 + 1)^2} = 0 \implies x^2 = 1 \implies x = \pm 1.
\]

Consider any \( 0 < h < 1 \). Then

\[
f'(-1 - h) = \frac{3 - 3(-1 - h)^2}{((-1 - h)^2 + 1)^2} = \frac{3 - 3(1 + h)^2}{((1 + h)^2 + 1)^2} < 0
\]

and

\[
f'(-1 + h) = \frac{3 - 3(-1 + h)^2}{((-1 + h)^2 + 1)^2} = \frac{3 - 3(1 - h)^2}{((1 - h)^2 + 1)^2} > 0.
\]

Thus by First Derivative Test, \( f \) has a local minimum at -1. Further

\[
f'(1 - h) = \frac{3 - 3(1 - h)^2}{((1 - h)^2 + 1)^2} > 0
\]
and
\[ f'(1 + h) = \frac{3 - 3(1 + h)^2}{(1 + h)^2 + 1} < 0. \]

Thus by First Derivative Test, \( f \) has a local maximum at 1.

Note that \( |f(x)| = \frac{3}{|x + 1/x|} \), for all \( x \neq 0 \). Clearly \( |f(0)| = 0 < 3/2 \). We now prove that for all \( x \neq 0 \),
\[
\frac{-3}{2} \leq \frac{3}{x + 1/x} \leq \frac{3}{2} \iff \left| x + \frac{1}{x} \right| \geq 2.
\]

Consider \( x > 0 \). Then clearly \( x + 1/x > 0 \). Suppose \( 0 < x + 1/x < 2 \). This gives \( 0 < x^2 + 1 < 2x \), that is, \( x^2 - 2x + 1 < 0 \), which says \( (x - 1)^2 < 0 \). This is not possible. Thus \( x + 1/x \geq 2 \). Now consider \( x < 0 \). Then \( -x > 0 \) and we get \( -x - 1/x < 2 \). Since the sign of \( x \) and \( x + 1/x \) are always the same, combining the above observations, we get \( |x + 1/x| \geq 2 \).

(b) \( f(x) = x + \frac{1}{x} \)

Solution. We have
\[ f'(x) = 1 - \frac{1}{x^2} \quad \text{and} \quad f''(x) = \frac{2}{x^3}. \]

So
\[ f'(x) = 0 \iff \frac{1}{x^2} = 1 \iff x^2 = 1 \iff x = \pm 1. \]

Further \( f''(-1) = -2 < 0 \) and \( f''(1) = 2 > 0 \). So by second derivative test, \( f \) has a local maximum at \(-1\) and a local minimum at \(1\). Also \( f(-1) = -2 \) and \( f(1) = 2 \). Since \( f(-1) < f(1) \), we conclude that \(-1\) is not a point of global maximum and \(1\) is not a point of global minimum. Also since, on \( \mathbb{R} \), every point of global maximum (minimum) is a point of local maximum (minimum), \( f \) has no global maxima or minima.

(2) [3 marks] (Incorrect question. 3 marks will be awarded by default to all students.)

Let the demand curve of a firm be \( P = f(Q) \), where \( f \) is differentiable with \( f'(Q) \neq 0 \). Let the maximum of the total revenue of the firm be at a price \( P_0 \). Show that \( \epsilon_d(P_0) = -1 \).

(3) [3 marks]

The average cost of a firm is
\[ A(Q) = 15 - 6Q + Q^2 + \frac{1}{Q}, \]

where \( Q \) is the input. Find the total cost and the marginal cost curves.

Solution. The average cost is \( A(Q) = \frac{T(Q)}{Q} \), where \( T(Q) \) is the total cost. So the total cost is \( T(Q) = QA(Q) = Q^3 - 6Q^2 + 15Q + 1 \). The marginal cost is \( M(Q) = T'(Q) = 3Q^2 - 12Q + 15 \).

(4) [4 marks]

The demand function for a goods is given by the relation \( P = 50 - 2Q \), while the total cost is given by \( TC = 16 + 2Q \).

(a) Find appropriate relations and compare the graphs of total profit with total cost and find the break-even points for the firm.

Solution. The total revenue function is
\[ TR(Q) = PQ = 50Q - 2Q^2. \]
So the total profit function is
\[ TP(Q) = TR(Q) - TC(Q) = 50Q - 2Q^2 - 16 - 2Q = 48Q - 2Q^2 - 16. \]

So the break-even is achieved when
\[ TP(Q) = 0 \iff 2Q^2 - 48Q + 16 = 0 \iff Q^2 - 24Q + 8 = 0 \iff Q = \frac{24 \pm \sqrt{576 - 32}}{2} = \frac{24 \pm \sqrt{544}}{2}. \]

Thus the break-even points are \( Q = 12 \pm \sqrt{136}. \)

(b) Compare the levels at which profit and revenue are maximized.

**Solution.** We have \((TP)'(Q) = 48 - 4Q\). So
\[ (TP)'(Q) = 0 \iff 48 - 4Q = 0 \iff Q = 12. \]

Further \((TP)''(Q) = -4\) and so \((TP)'(12) = -4 < 0\). Thus the total profit is maximized at \( Q = 12 \).

Now \((TR)'(Q) = 50 - 4Q\). So
\[ (TR)'(Q) = 0 \iff 50 - 4Q = 0 \iff Q = \frac{25}{2} . \]

Further \((TR)''(Q) = -4\) and so \((TR)''(25/2) = -4 < 0\). Thus the revenue is maximized when \( Q = 25/2 \).

(5) [6 marks]

Prove the following relations.

(a) \( MR = P(1 + \frac{1}{\epsilon_d}) \).

**Solution.** We consider the price \( P \) to be a function of quantity \( Q \). Thus we have the total revenue \( TR = PQ \). So
\[ MR = \frac{d(TR)}{dQ} = Q \frac{dP}{dQ} + P = P \left( \frac{Q}{P} \cdot \frac{dP}{dQ} + 1 \right) = P \left( \frac{1}{\epsilon_d} + 1 \right). \]

(b) If profit is maximum at \( Q_0 \), then \( MR = MC \) at \( Q_0 \).

**Solution.** We consider the total revenue \( TR \) to be a function of the quantity \( Q \). So the profit function is \( \Pi = TR - TC \). If the profit is maximum at \( Q_0 \), we get
\[ \frac{d\Pi}{dQ}(Q_0) = 0 \iff \frac{d(TR)}{dQ}(Q_0) - \frac{d(TC)}{dQ}(Q_0) = 0 \iff MR(Q_0) - MC(Q_0) = 0. \]

(c) \( MC = AC_{\min} \).

**Solution.** We consider the total cost \( TC \) to be a function of the quantity \( Q \). Then \( TC = (AC)Q \). This gives
\[ MC = Q \frac{d(AC)}{dQ} + AC. \]

So if \( Q_0 = AC_{\min} \), then \( \frac{d(AC)}{dQ}(Q_0) = 0 \) and hence \( MC(Q_0) = AC(Q_0) = AC_{\min} \).