Measure (Lectures 5, 6, 7, 8 and 9)

2.1. Set functions

(2.1) Let $X$ be any countably infinite set and let
\[ C = \{ \{x\} \mid x \in X \}. \]
Show that the algebra generated by $C$ is
\[ \mathcal{F}(C) := \{ A \subseteq X \mid A \text{ or } A^c \text{ is finite} \}. \]

Let $\mu : \mathcal{F}(C) \rightarrow [0, \infty)$ be defined by
\[ \mu(A) := \begin{cases} 
0 & \text{if } A \text{ is finite,} \\
1 & \text{if } A^c \text{ is finite.}
\end{cases} \]

Show that $\mu$ is finitely additive but not countably additive. If $X$ is an
uncountable set, show that $\mu$ is also countably additive.

(2.2) Let $X = \mathbb{N}$, the set of natural numbers. For every finite set $A \subseteq X$, let
\( \#A \) denote the number of elements in $A$. Define for $A \subseteq X$,
\[ \mu_n(A) := \frac{\#\{ m : 1 \leq m \leq n, m \in A \}}{n}. \]

Show that $\mu_n$ is countably additive for every $n$ on $\mathcal{P}(X)$. In a sense, $\mu_n$ is
the proportion of integers between 1 to $n$ which are in $A$. Let
\[ C = \{ A \subseteq X \mid \lim_{n \rightarrow \infty} \mu_n(A) \text{ exists} \}. \]

Show that $C$ is closed under taking complements, finite disjoint unions and
proper differences.
(2.3) Let \( \mu : \tilde{I} \cap (0, 1] \rightarrow [0, \infty] \) be defined by
\[
\mu(a, b) := \begin{cases} 
  b - a & \text{if } a \neq 0, 0 < a < b \leq 1, \\
  +\infty & \text{otherwise}.
\end{cases}
\]
(Recall that \( \tilde{I} \cap (0, 1] \) is the class of all left-open right-closed intervals in \( (0, 1] \).)
Show that \( \mu \) is finitely additive. Is \( \mu \) countably additive also?

(2.4) Let \( \mathcal{A} \) be an algebra of subsets of a set \( X \).
(i) Let \( \mu_1, \mu_2 \) be measures on \( \mathcal{A} \), and let \( \alpha \) and \( \beta \) be nonnegative real numbers. Show that \( \alpha \mu_1 + \beta \mu_2 \) is also a measure on \( \mathcal{A} \).
(ii) For any two measures \( \mu_1, \mu_2 \) on \( \mathcal{A} \), we say \( \mu_1 \leq \mu_2 \) if \( \mu_1(E) \leq \mu_2(E), \forall E \in \mathcal{A} \).
Let \( \{\mu_n\}_{n \geq 1} \) be a sequence of measures on \( \mathcal{A} \) such that
\[
\mu_n \leq \mu_{n+1}, \forall n \geq 1.
\]
Define \( \forall E \in \mathcal{A} \),
\[
\mu(E) := \lim_{n \rightarrow \infty} \mu_n(E).
\]
Show that \( \mu \) is also a measure on \( \mathcal{A} \) and \( \forall E \in \mathcal{B} \),
\[
\mu(E) = \sup \{\mu_n(E) \mid n \geq 1\}.
\]
(2.5) Let \( X \) be a compact topological space and \( \mathcal{A} \) be the collection of all those subsets of \( X \) which are both open and closed. Show that \( \mathcal{A} \) is an algebra of subsets of \( X \). Further, every finitely additive set function on \( \mathcal{A} \) is also countably additive.

Optional Exercises

(2.6) Let \( X \) be a nonempty set.
(a) Let \( \mu : \mathcal{P}(X) \rightarrow [0, \infty) \) be a finitely additive set function such that \( \mu(A) = 0 \) or 1 for every \( A \in \mathcal{P}(X) \). Let \( \mathcal{U} = \{A \in \mathcal{P}(X) \mid \mu(A) = 1\} \).
Show that \( \mathcal{U} \) has the following properties:
(i) \( \emptyset \notin \mathcal{U} \).
(ii) If \( A \in X \) and \( B \supseteq A \), then \( B \in \mathcal{U} \).
(iii) If \( A, B \in \mathcal{U} \), then \( A \cap B \in \mathcal{U} \).
(iv) For every \( A \in \mathcal{P}(X) \), either \( A \in \mathcal{U} \) or \( A^c \in \mathcal{U} \).
(Any \( \mathcal{U} \subseteq \mathcal{P}(X) \) satisfying (i) to (iv) is called an \textbf{ultrafilter} in \( X \).)
(b) Let \( \mathcal{U} \) be any ultrafilter in \( X \). Define \( \mu : \mathcal{P}(X) \rightarrow [0, \infty) \) by
\[
\mu(A) := \begin{cases} 
  1 & \text{if } A \in \mathcal{U}, \\
  0 & \text{if } A \notin \mathcal{U}.
\end{cases}
\]
Show that \( \mu \) is finitely additive.
2.2. Countably additive set functions on intervals

(2.7) Let \( F(x) = [x] \), the integral part of \( x, x \in \mathbb{R} \). Describe the set function \( \mu_F \) induced by \( F \) on the class \( \mathcal{I} \) of all left-open right-closed intervals.

(2.8) Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be a distribution function and \( \alpha \in \mathbb{R} \). Show that \( F_1 := F + \alpha \) is also a distribution function and \( \mu_F = \mu_{F_1} \). Is the converse true?

(2.9) (i) Let \( C \) be a collection of subsets of a set \( X \) and \( \mu : C \rightarrow [0, \infty] \) be a set function. If \( \mu \) is a measure on \( C \), show that \( \mu \) is finitely additive. Is \( \mu \) monotone? Countably subadditive?

(ii) If \( C \) be a semi-algebra, then \( \mu \) is countably subadditive iff \( \forall A \in C \) with \( A \subseteq \bigcup_{i=1}^{\infty} A_i, A_i \in C \) implies

\[
\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i).
\]

2.3. Set functions on algebras

(2.10) Let \( A \) be an algebra of subsets of a set \( X \) and \( \mu : A \rightarrow [0, \infty] \) be a finitely additive set function.

(i) Show that in general, for a decreasing sequence \( \{A_k\}_{k \geq 1} \) in \( A \) with \( \bigcap_{k=1}^{\infty} A_k = A \in A \) need not imply that \( \mu(A) = \lim_{n \to \infty} \mu(A_n) \), even if \( \mu \) is countably additive.

(ii) If \( \mu(X) < +\infty \), show that the following statements are equivalent:

(a) \( \lim_{k \to \infty} \mu(A_k) = 0 \), whenever \( \{A_k\}_{k \geq 1} \) is a sequence in \( A \) with \( A_k \supseteq A_{k+1} \forall k, \) and \( \bigcap_{k=1}^{\infty} A_k = \emptyset \).

(b) \( \mu \) is countably additive.

(2.11) Let \( A \) be a \( \sigma \)-algebra and \( \mu : A \rightarrow [0, \infty] \) be a measure. For any sequence \( \{E_n\}_{n \geq 1} \) in \( A \), show that

(i) \( \mu \left( \lim_{n \to \infty} \inf E_n \right) \leq \lim_{n \to \infty} \inf \mu(E_n) \).

(ii) \( \mu \left( \lim_{n \to \infty} \sup E_n \right) \geq \lim_{n \to \infty} \sup \mu(E_n) \).

(Hint: For a sequence \( \{E_n\}_{n \geq 1} \) of subsets of a set \( X \),

\[
\liminf_{n \to \infty} E_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k \quad \leq \quad \limsup_{n \to \infty} E_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.
\]
Optional Exercise

(2.12) Let $\mathcal{A}$ be a semi-algebra of subsets of a set $X$ and $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive set function. Show that the following statements are equivalent:

(a) $\lim_{k \to \infty} \mu(A_k) = 0$, whenever $\{A_k\}_{k \geq 1}$ is an increasing sequence in $\mathcal{A}$ with $\bigcup_{k=1}^{\infty} A_k = A \in \mathcal{A}$.

(b) $\mu$ is countably additive.

(Hint: Extend $\mu$ to the algebra generated by $\mathcal{A}$.)

2.4. Uniqueness problem for measures

(2.13) Let $\mathcal{A}$ be an algebra of subsets of a set $X$. Let $\mu_1$ and $\mu_2$ be $\sigma$-finite measures on a $\sigma$-algebra $\mathcal{S}(\mathcal{A})$ such that $\mu_1(A) = \mu_2(A) \ \forall \ A \in \mathcal{A}$. Then, $\mu_1(A) = \mu_2(A) \ \forall A \in \mathcal{S}(\mathcal{A})$.

(2.14) Show that a measure $\mu$ defined on an algebra $\mathcal{A}$ of subsets of a set $X$ is finite if and only if $\mu(X) < +\infty$. 