Lecture no. 18

Measure & Integration

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$$s_n \uparrow$$
$$s_n' \uparrow \quad \text{and} \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_n'$$

<u>Fix</u> any <u>$m \geq 1$</u>, and consider

$$s_n \wedge s_m', \quad n \geq 1$$

$$s_n \wedge s_m' \leq s_n \;\; \forall n$$

also as $n \longrightarrow \infty$, $s_n \wedge s_m' \longrightarrow s_m'$

$$\Rightarrow \int s_n \, d\mu \geq \int (s_n \wedge s_m') \, d\mu \longrightarrow \int s_m' \, d\mu$$

$$\int s_n \, d\mu \geq \int s_m' \, d\mu \;\; \forall n$$

$$\Rightarrow \quad \lim_{n\to\infty} \int s_n \, d\mu \geq \int s_m' \, d\mu$$

$$\Longrightarrow \quad \underline{\lim_{n\to\infty} \int s_n \, d\mu} \geq \underline{\lim_{m\to\infty} \int s'_m \, d\mu}$$

Similarly

$$\Longrightarrow \quad \lim_{m\to\infty} \int s'_m \, d\mu \geq \lim_{n\to\infty} \int s_n \, d\mu$$

$$\Longrightarrow \quad \lim_{m\to\infty} \int s'_m \, d\mu = \lim_{n\to\infty} \int s_n \, d\mu$$

$\square$

$s \in \mathbb{L}^+_0, \quad \int s \, d\mu$

$\mathbb{L}^+_0 \subseteq \mathbb{L}^+, \quad s \in \mathbb{L}^+ =$

$s_n = s \ \forall \ n$

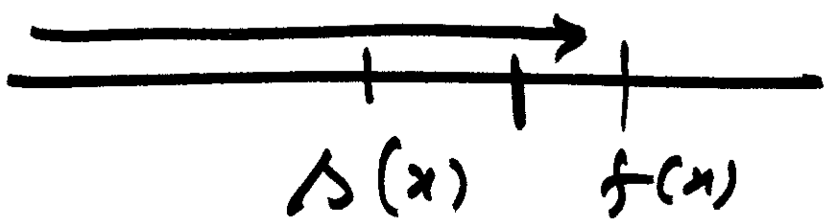
$\Longrightarrow \int s \, d\mu = \int s \, d\mu$

$$s \in \mathbb{L}_0^+, \quad f \in \mathbb{L}^+$$

$$0 \leq s \leq f$$

$$f \in \mathbb{L}^+ \Longrightarrow \exists\, s_n \in \mathbb{L}_0^+$$

$$s_n \uparrow f \text{ and } \int s_n \, d\mu \to \int f \, d\mu.$$



Define $B_n = \{ x \in X \mid s(x) \leq s_n(x) \}$

$$B_n \in \mathcal{S}, \quad B_n \uparrow X$$

$$\int s\,d\mu = \lim_{n\to\infty} \int_{B_n} s\,d\mu$$

$$\leq \lim_{n\to\infty} \int_{B_n} s_n\,d\mu$$

$$\leq \lim_{n\to\infty} \int s_n\,d\mu$$

$$= \int f\,d\mu.$$

$$\Rightarrow \quad \int s\,d\mu \leq \int f\,d\mu, \quad 0 \leq s \leq f$$

let $\beta = \sup\{ \int s \, d\mu \mid 0 \leq s \leq f \}$

To show $\int f \, d\mu = \beta$.

$f \in \mathbb{I}^+ \Rightarrow s_n \in \mathbb{I}_0^+, \quad s_n \uparrow f$

$$\int f \, d\mu = \lim_{n \to \infty} \int s_n \, d\mu.$$

<u>In case</u> $\beta = +\infty$, then $\forall$ N, $\exists$ $s \in \mathbb{I}_0^+$ s.t. $s \leq f$

$$\int s \, d\mu \geq N$$

$$\Rightarrow \quad s_n \uparrow f$$

let
$$\beta = \sup\left\{ \int s \, d\mu \;\middle|\; \begin{array}{l} 0 \le s \le f \\ s \in \mathbb{I}_0^+ \end{array} \right\}$$

To show $\beta = \int f \, d\mu$.

<u>In case</u> $\int f \, d\mu = +\infty$,

$f \in \mathbb{I}^+$, $\exists\, s_n \in \mathbb{I}_0^+$, $s_n \uparrow f$

$$\int s_n \, d\mu \longrightarrow \int f \, d\mu = +\infty$$

$\forall\, N$, $\exists\, n_0$ s.t $\int s_{n_0} \, d\mu \geq N$

$s_{n_0} \le f$

$$\Rightarrow \quad \beta \geq N \quad \forall N$$

$$\Rightarrow \quad \beta = +\infty$$

$$\beta = +\infty = \int f \, d\mu.$$

In case $\int f \, d\mu < +\infty$, $\int s_n \, d\mu \to \int f d\mu$

$\forall \; \varepsilon > 0$, $\exists \; n_0$ such that

$$\int f d\mu - \int s_{n_0} d\mu < \varepsilon$$

$$\underline{\int f d\mu} \leq \int s_{n_0} d\mu + \varepsilon \leq \underline{\beta + \varepsilon}$$

$$\Rightarrow \quad \int f d\mu \leq \beta.$$

$$\beta \leq \int f\,d\mu$$

$$\Longrightarrow \quad \beta = \int f\,d\mu.$$

$$\forall\ f \in \mathbb{L}^+, \quad \int f\,d\mu$$

$$f \in \mathbb{L}^+ \Longrightarrow s_n \in \mathbb{L}^+_0,\ s_n \uparrow f$$

$$\lim_{n\to\infty} \int s_n\,d\mu = \int f\,d\mu$$

$$g \in \mathbb{L}^+ \Longrightarrow s'_n \in \mathbb{L}^+_0,\ s'_n \uparrow g$$

$$\lim_{n\to\infty} \int s'_n\,d\mu = \int g\,d\mu$$

$\Longrightarrow \quad \alpha s_n \uparrow \alpha f$

and $\displaystyle \lim_{n\to\infty} \int (\alpha s_n)\, d\mu$

$\displaystyle = \alpha \left(\lim_{n\to\infty} \int s_n \, d\mu \right)$

$\displaystyle = \alpha \int f \, d\mu.$

$\displaystyle \lim_{n\to\infty} \int \beta s_n' \, d\mu = \beta \int g \, d\mu$

$\left(\cancel{\alpha s_n + \beta g_n} \right) \qquad \underline{\alpha s_n + \beta s_n'} \uparrow \alpha f + \beta g$

$\cancel{\int \alpha s_n d}$

$$\lim_{n \to \infty} \int (\alpha s_n + \beta s_n') \, d\mu = \underline{\int (\alpha f + \beta g) \, d\mu}$$

$$\| $$

$$\lim_{n \to \infty} \left[\alpha \int s_n \, d\mu + \beta \int s_n' \, d\mu \right]$$

$$\| $$

$$\alpha \left(\lim_{n \to \infty} \int s_n \, d\mu \right) + \beta \left(\lim_{n \to \infty} \int s_n' \, d\mu \right)$$

$$\| $$

$$\underline{\alpha \int f \, d\mu} + \underline{\beta \int g \, d\mu}$$

$$f \in \mathbb{L}^+ \Longrightarrow \exists\, s_n \in \mathbb{L}^+_0$$

$s_n \uparrow f$ and $\displaystyle\int f\,d\mu = \lim_{n\to\infty} \int s_n\,d\mu$

Now $E \in \mathcal{S}$, $\underbrace{\chi_E s_n} \uparrow \underbrace{\chi_E f}$

$\Longrightarrow \chi_E f \in \mathbb{L}^+$ and

$$\int \chi_E f\,d\mu = \lim_{n\to\infty} \int \chi_E s_n\,d\mu$$

<u>Claim</u> $\displaystyle \nu(E) = \int_E f\,d\mu.$

ν is a measure.

To prove:

$$E = \bigsqcup_{i=1}^{\infty} E_i, \quad E_i \in S$$

Then

$$\nu(E) = \sum_{i=1}^{\infty} \nu(E_i) \ ?$$

$$\begin{aligned}
\nu(E) &:= \int \chi_E f \, d\mu \\
&= \lim_{n \to \infty} \left(\int \chi_E s_n \, d\mu \right) \\
&= \lim_{n \to \infty} \left(\sum_{i=1}^{\infty} \int \chi_{E_i} s_n \, d\mu \right) \\
&= \sum_{i=1}^{\infty} \left(\lim_{n \to \infty} \int \chi_{E_i} s_n \, d\mu \right)
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \cancel{\lim_{n\to\infty}} \int \chi_{E_i} f \, d\mu$$

$$= \sum_{i=1}^{\infty} \nu(E_i)$$

Suppose $\mu(E) = 0$

$$\nu(E) = \int \chi_E f \, d\mu$$

$$= \lim_{n\to\infty} \int \underline{\chi_E s_n} \, d\mu$$

$$= 0$$

$$N = \{ x \in X \mid f_1(x) \neq f_2(x) \}$$

$$X = N \cup N^c$$

Given $\cancel{f_1(N^c) = 0}$ $\mu(N) = 0$.

$$\int f_1 \, d\mu = \int_N f_1 \, d\mu + \int_{N^c} f_1 \, d\mu$$

$$= 0 + \int_{N^c} f_1 \, d\mu$$

$$= \int_N f_2 \, d\mu + \int_{N^c} f_2 \, d\mu$$

$$= \int f_2 \, d\mu$$