Lecture 15

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Measure & Integration

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let $\quad f : X \longrightarrow \mathbb{R}$

such that $\exists \{s_n\}_{n \geq 1}$ of simple, mble non-negative functions $\underline{s_n \uparrow f}$

$$f(x) = \lim_{n \to \infty} s_n(x) \quad \forall x \in X.$$

<u>To show</u> f is <u>measurable.</u>

let $c \in \mathbb{R}$

$\cancel{f^{-1}[c, +\infty] = \{x \in X \mid f(x) > c}$

$$f^{-1}[-\infty, c] = \{x \in X \mid f(x) \leq c\}$$

$$f(x) = \lim_{n \to \infty} s_n(x) \quad , \quad s_n(x) \uparrow$$

$$\Rightarrow \quad s_n(x) \leq c \quad \forall \; n \geq 1$$

$$\Rightarrow \quad f^{-1}[-\infty, c] \subseteq \bigcap_{n=1}^{\infty} \{x : s_n(x) \leq c\}$$

If $s_n(x) \leq c \quad \forall \; n$

$$\Rightarrow \quad f(x) \leq c$$

Hence

$$f^{-1}[-\infty, c] = \bigcap_{n=1}^{\infty} \{x : s_n(x) \leq c\}$$

$$= \bigcap_{n=1}^{\infty} \underbrace{s_n^{-1}[-\infty, c]}_{\cdot}$$

$$\Rightarrow \quad f^{-1}[-\infty, c] \in \mathcal{S}.$$

Hence f is measurable.

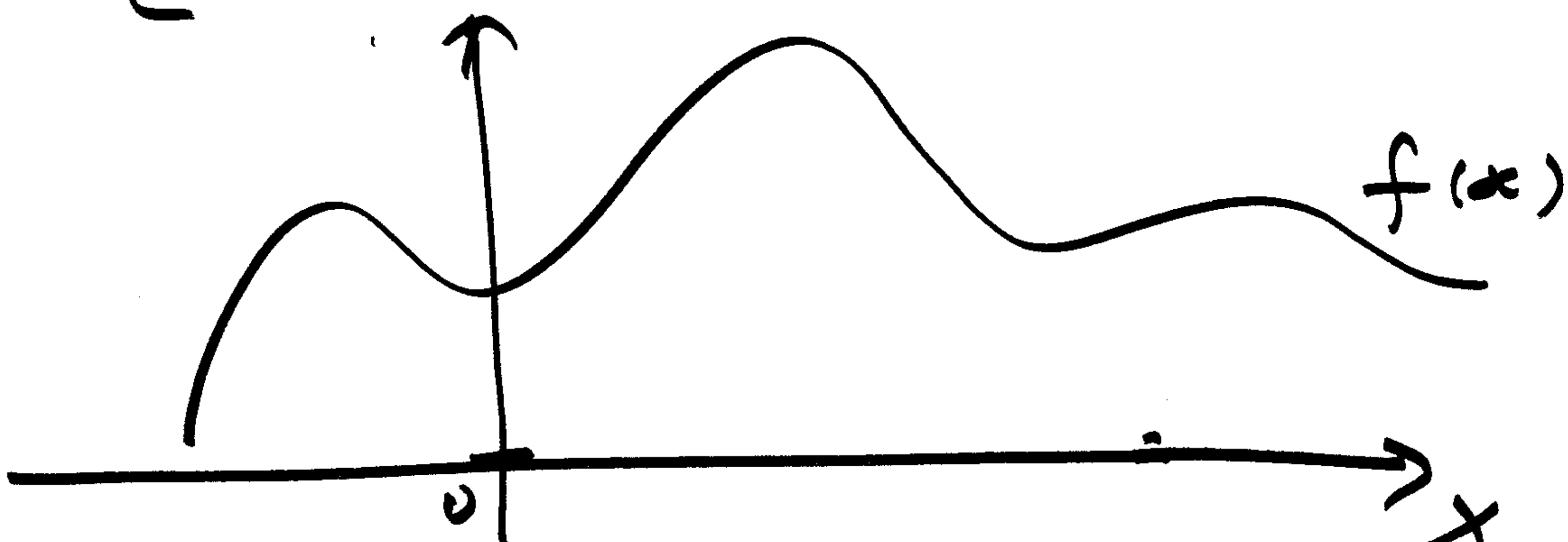
<u>Conversely</u> let $f : X \longrightarrow [0, +\infty]$

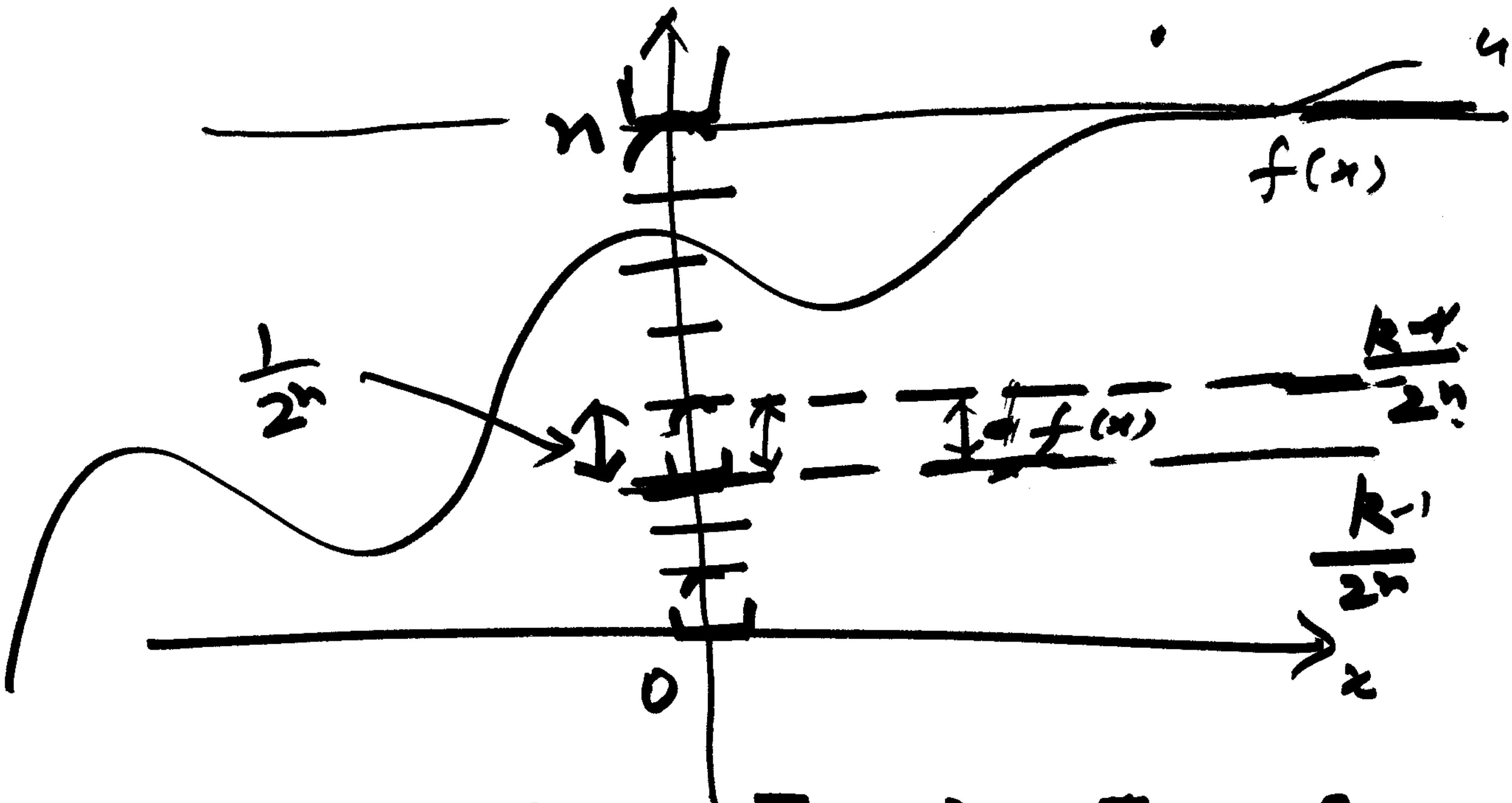
be measurable.

To construct a sequence $\{s_n\}_{n \geq 1}$

s_n is simple, nonnegative

?? $s_n \uparrow f$





$$[0,+\infty] = [0,n) \cup [n,+\infty]$$

$$= \bigcup_{k=1}^{n\cdot 2^n} \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \cup [n,+\infty)$$

For $x \in X$, $f(x) \in [0, \infty]$ 5

$$\Rightarrow f(x) \in \left(\bigcup_{k=1}^{n \cdot 2^n} \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right) \right) \cup [n, +\infty]$$

$\Rightarrow$ Either $f(x) \in [n, +\infty)$

or $f(x) \in \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right)$

Define $s_n(x) = \begin{cases} n & \text{if } f(x) \in [n, +\infty] \\ \frac{k-1}{2^n} & \text{if } f(x) \in \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right) \end{cases}$

Claim This is the required sequence

Note

$$s_n(x) = n \cdot \chi_{f^{-1}[n,+\infty]}(x) + \sum_{k=1}^{n2^n} \chi_{f^{-1}\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x) \cdot \left(\frac{k-1}{2^n}\right)$$

$\Longrightarrow$ s_n is nonnegative simple measurable

Show s_n is increasing.

Fix $x \in X$

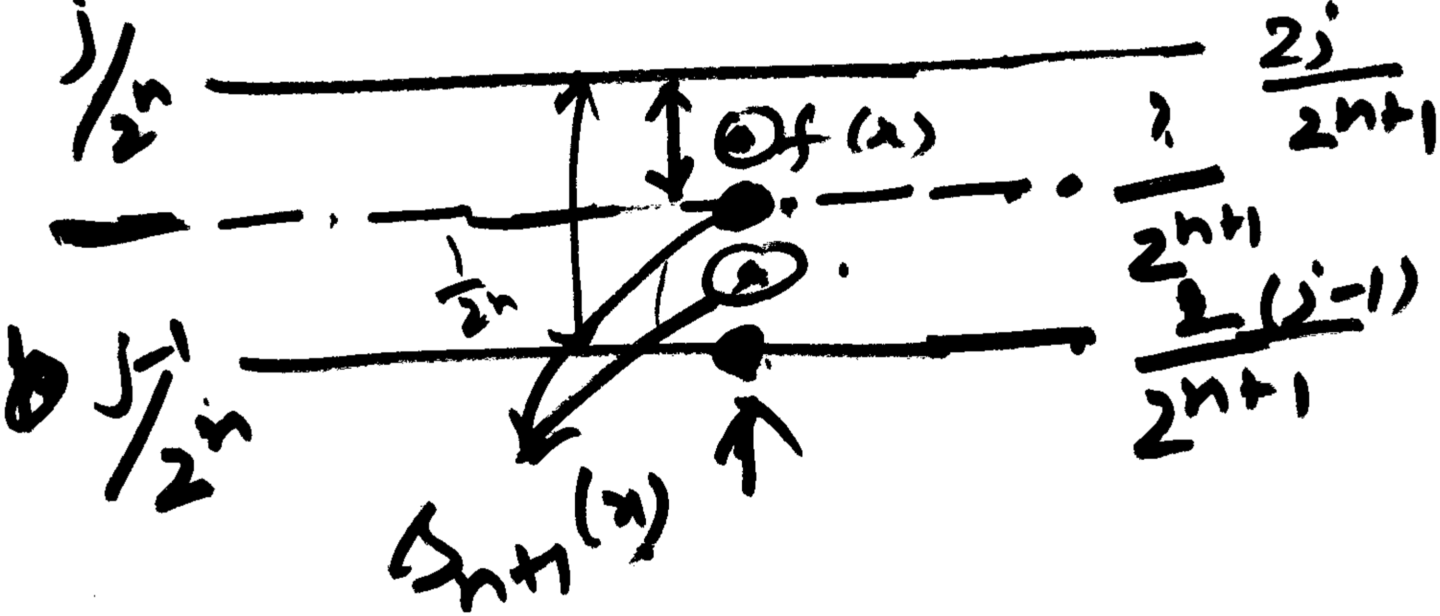
To show $s_{n+1}(x) \geq s_n(x) \;\forall n$

~~Either~~

~~$s \to f(x) \geq n+1$~~

Fix n,

$$s_{n+1}(x) = \begin{cases} n+1 \\ \text{or} \\ \dfrac{k-1}{2^{n+1}} \text{ for some } 1 \le k \le \dfrac{(n+1)}{2^{n+1}} \end{cases}$$



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$$s_{n+1}(x) = \begin{cases} n+1 > n = s_n(x) \\ \dfrac{k-1}{2^{n+1}} > \dfrac{j-1}{2^n} = s_n(x) \end{cases}$$

Hence $s_n(x) \uparrow$

$s_n(x) \longrightarrow f(x)$?

Fix x either $f(x) = +\infty$

But then $s_n(x) = n \longrightarrow +\infty$

or $0 \leq f(x) \leq n < n+1$ —

$f(x) \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \Rightarrow s_n(x) = \dfrac{k-1}{2^n}$

$$|f(x) - s_n(x)| < \frac{}{2^{n_0}} \text{ for some } n_0 \qquad 9$$

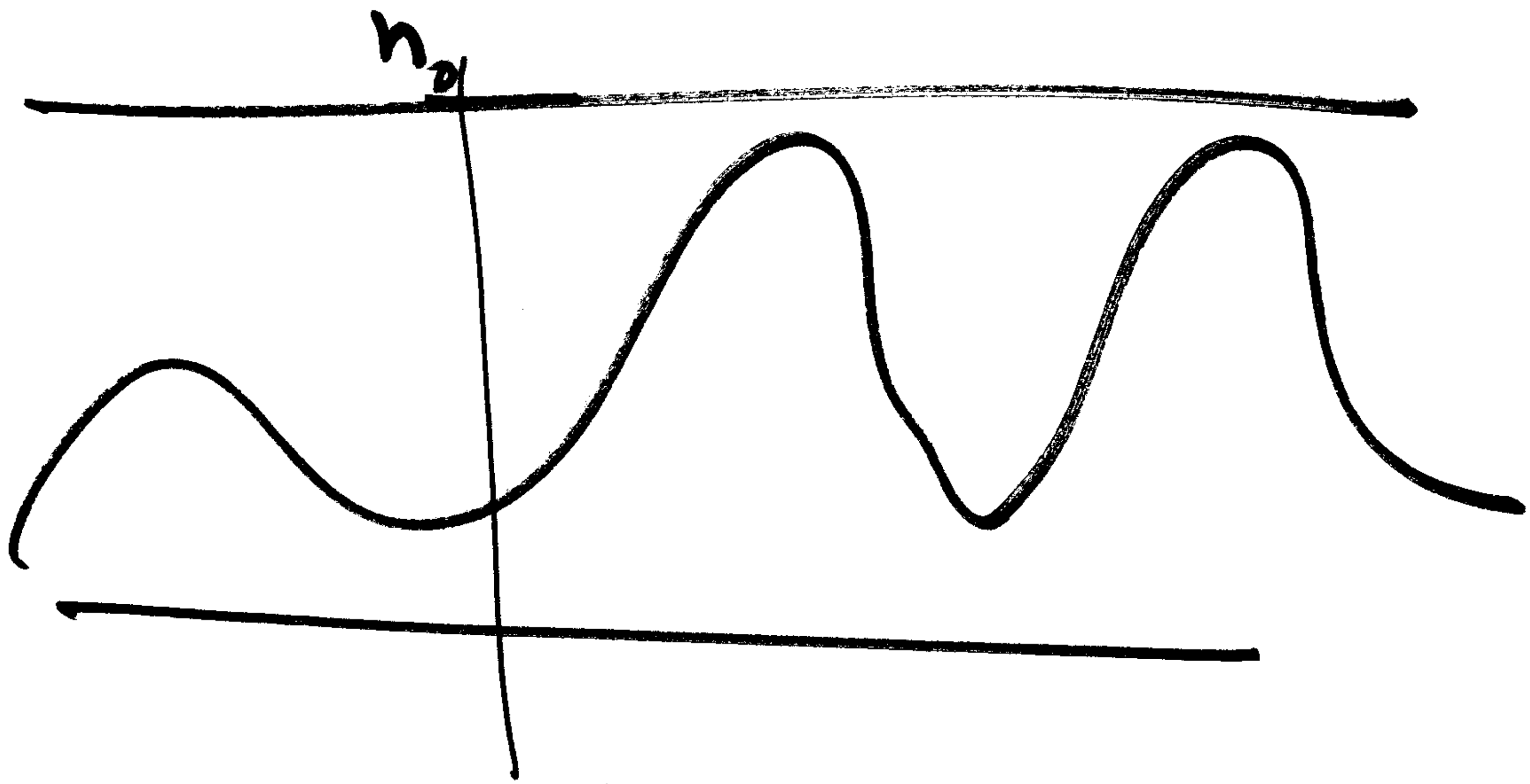
$\forall\, n \geq n_0$

$$|f(x) - s_n(x)| < \frac{1}{2^n} \quad \forall n \geq n_0$$

$$\Longrightarrow \quad s_n(x) \longrightarrow f(x)$$

$$f : X \longrightarrow [0, +\infty] \quad \text{mble}$$

$\exists\ 0 \leq s_n \uparrow f$



$$f(x) \in [0, n)$$

$$|f(x) - s_n(x)| < \frac{1}{2^n} \quad \forall\, n \geq n_0$$

f mble

$$\begin{tikzcd} \underline{f} = & \underline{f^+} & - & \underline{f^-} & \\ & s_n \arrow[u] & - & s_n' \arrow[u] \arrow[r] & f \\ & & \phi_n \arrow[urr] & & \end{tikzcd}$$

f measurable

$$\Longrightarrow \exists\, s_n \longrightarrow f$$

$$\Longrightarrow \alpha s_n \longrightarrow f$$

$$\Longrightarrow f \text{ is measurable}$$

$$\begin{array}{ccc} f & & g \\ \uparrow & & \uparrow \\ s_n & & s_n' \end{array}$$

$$\Longrightarrow s_n + s_n' \longrightarrow f+g$$

$$\Longrightarrow f+g \text{ is measurable.}$$

$|f| = f^+ + f^-$

f mble $\Rightarrow f^+, f^-$ measurable

$\Rightarrow f^+ + f^-$ measurable

$\quad\quad\quad\;\, \| $

$\quad\quad\quad\;\, f$

or

f measurable

$\Rightarrow \exists\, s_n \longrightarrow f$

$\underline{|s_n|} \longrightarrow \underline{|f|}$

$\Rightarrow |f|$ is measurable.

f measurable

$\Longrightarrow \quad \exists\ s_n \longrightarrow f$

$\Longrightarrow \quad \underline{\chi_E \cdot s_n} \longrightarrow \chi_E f$

$\Longrightarrow \quad \chi_E f$ is measurable

$f, \ g$ are measurable

$\uparrow \qquad \uparrow \quad \Longrightarrow \quad \underline{s_n \cdot s_n'} \longrightarrow fg$

$s_n \qquad s_n' \quad \Longrightarrow \quad fg$ is measurable