Lecture 13

I. K Rana

Measure and Integration

20/12/10

$(\mathbb{R}, \mathcal{L}, \lambda)$ – lebesgue measure space

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$$

$\mathbb{R}, +$

$$E \subseteq \mathbb{R}, \quad E + x := \{ y + x \mid y \in E \}$$

<u>Question</u>

$$\left. \begin{aligned} E \in \mathcal{L} &\Longrightarrow E + x \in \mathcal{L} \ ? \\ E \in \mathcal{B}_{R} &\Longrightarrow E + x \in \mathcal{B}_{\mathbb{R}} \ ? \end{aligned} \right\|$$

$$\lambda^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \lambda(I_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$$

I_j's <u>open</u>

$E \in \mathcal{L}$. To show $E+x \in \mathcal{L}$?

To show $\forall\ Y \subseteq \mathbb{R}$

$$\lambda^*(Y) = \lambda^*(Y \cap (E+x)) + \lambda^*(Y \cap (E+x)^c)$$

$E \in \mathcal{L}$ $\implies$ $\forall\ Y \subseteq \mathbb{R}$

$$\lambda^*(Y) = \lambda^*(Y \cap E) + \lambda^*(Y \cap E^c)$$

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$$E \subseteq \bigcup_{j=1}^{\infty} I_j \iff E+x \subseteq \bigcup_{j=1}^{\infty} (I_j+x)$$

$$\implies \quad \lambda^*(E) = \lambda^*(E+x)$$

$E \in \mathcal{L}$

$$\Longrightarrow \quad \lambda^*(Y) = \lambda^*(Y \cap E) + \lambda^*(Y \cap E^c)$$

$$= \lambda^*((Y \cap E) + x) + \lambda^*((Y \cap E^c) + x)$$

$$= \lambda^*[(Y+x) \cap (E+x)] + \lambda^*[(Y+x) \cap (E^c+x)]$$

by $Y-x$

$$\Longrightarrow \quad \begin{array}{ccc} \lambda^*(Y-x) = & \lambda^*(Y \cap (E+x)) + & \lambda^*[Y \cap (E^c+x)] \\ \Vert & & \Vert \\ \lambda^*(Y) \quad = & \lambda^*(Y \cap (E+x)) + & \lambda^*(Y \cap (E+x)^c) \end{array}$$

$$\Longrightarrow \quad E + x \in \mathcal{L}.$$

$$E \in \mathcal{B}_{\mathbb{R}} \Longrightarrow E + x \in \mathcal{B}_{\mathbb{R}} ?$$

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ y & \longmapsto & y + x \quad \forall y \in Y \end{array}$$

Observation: Translation is a homeomorphis: one-one / onto / both ways continuous

Consider

$$S = \{ E \in \mathcal{B}_{\mathbb{R}} \mid E + x \in \mathcal{B}_{\mathbb{R}} \}$$

(i) $\mathcal{O} \subseteq S$

(ii) S is a σ-algebra

If $E \subseteq \mathbb{R}$ is open, then

$E+x$ is also an open set..

$$\Longrightarrow \quad \mathcal{O} \subseteq \mathcal{S}$$

<u>Note</u> (i) $\phi, \mathbb{R} \in \mathcal{S}$

(ii) $E \in \mathcal{S} \Longrightarrow E+x \in \mathcal{B}_{\mathbb{R}}$

$$\Longrightarrow (E+x)^c \in \mathcal{B}_{\mathbb{R}}$$

$$\Longrightarrow E^c+x \in \mathcal{B}_{\mathbb{R}}$$

$$\Longrightarrow E^c \in \mathcal{S}$$

(iii) let $E_n \in \mathcal{S}$, $n \geq 1$.

$$\Rightarrow \quad E_n + x \in \mathcal{B}_{\mathbb{R}}$$

$$\Rightarrow \quad \bigcup_n (E_n + x) \in \mathcal{B}_{\mathbb{R}}$$

$$\Rightarrow \quad \left(\bigcup_{n=1}^{\infty} E_n\right) + x \in \mathcal{B}_{\mathbb{R}}$$

$$\Rightarrow \quad \bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$$

Hence $\mathcal{O} \subseteq \mathcal{S}$, a σ-algebra

$$\Rightarrow \quad \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{S} \subseteq \mathcal{B}_{\mathbb{R}}$$

Suppose (i) holds: $E \in \mathcal{L}$

To show: (ii) $\forall\, \varepsilon > 0$, $\exists\, G_\varepsilon$ open such that $G_\varepsilon \supseteq E$ and $\lambda^*(G_\varepsilon \setminus E) < \varepsilon$?

Suppose E is such that $\lambda(E) < +\infty$

$$\lambda^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(I_j) \;\middle|\; \begin{array}{l} E \subseteq \bigcup I_j, \\ I_j \text{ open} \end{array} \right\}$$

Let $\varepsilon > 0$ be fixed. There exist I_j's s.t. $E \subseteq \bigcup_{j=1}^{\infty} I_j$, I_j open

and $\lambda^*(E) + \varepsilon > \sum_{j=1}^{\infty} \lambda(I_j)$

Note $\sum_{j=1}^{\infty} \lambda(I_j) < +\infty$

$$\Rightarrow \lambda^*\left(\bigcup_{j=1}^{\infty} I_j\right) \leq \sum_{j=1}^{\infty} \lambda^*(I_j) < +\infty$$

Put $G_\varepsilon := \bigcup_{j=1}^{\infty} I_j$

<u>Note</u> G_ε is open and $E \subseteq G_\varepsilon$.

and
$$\begin{aligned} \lambda^*(G_\varepsilon \setminus E) &= \lambda^*(G_\varepsilon) \setminus \lambda^*(E) \\ &= \lambda^*\left(\bigcup_{j=1}^{\infty} I_j\right) \setminus \lambda^*(E \\ &\leq \sum_{j=1}^{\infty} \lambda^*(I_j) \setminus \lambda^*(E) \\ &\qquad < \varepsilon \end{aligned}$$

Thus, if $\lambda^*(E) < +\infty$, then (i) $\Rightarrow$ (ii).

In general $E = \bigsqcup_{j=1}^{\infty} E_j$, $\lambda(E_j) < +\infty$.

By earlier case $\lambda^*(E_j) < +\infty$, fixed $\varepsilon > 0$, $\exists$ open set $G_j \supseteq E_j$ such that

$$\lambda^*(G_j \setminus E_j) < \varepsilon/2^j.$$

Define $G_\varepsilon = \bigcup_{j=1}^{\infty} G_j$. It is an open set and $G_\varepsilon \supset \bigcup_{j=1}^{\infty} E_j = E$

Further $G_\varepsilon \setminus E = \left(\bigcup_{j=1}^{\infty} G_j\right) \setminus \left(\bigcup_{j=1}^{\infty} E_j\right)$

$$\subseteq \bigcup_{j=1}^{\infty} (G_j \setminus E_j)$$

$$\lambda^*(G_\varepsilon \setminus E) \leq \sum_{j=1}^{\infty} \lambda^*(G_j \setminus E_j)$$

$$\leq \sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon.$$

<u>Hence</u> <u>(i) $\Longrightarrow$ (ii)</u>

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<u>(ii) $\Longrightarrow$ (iii)</u> Let $E \subseteq \mathbb{R}$ such that $\forall\ \varepsilon > 0$, $\exists$ open set $G_\varepsilon \supseteq E$ and

$$\overset{*}{\lambda}(G_\varepsilon \setminus E) < \varepsilon.$$

<u>In particular</u> $\forall\ \varepsilon = 1/n$, $\exists\ G_n \supseteq E$ G_n open such that

$$\lambda^*(G_n \setminus E) < \frac{1}{n}$$

Define $G = \bigcap_{n=1}^{\infty} G_n$. Note G is a G_δ-set and $G \supseteq E$, further

$$\underline{\lambda^*(G \setminus E)} \leq \overset{*}{\lambda}(G_n \setminus E) < \underline{\frac{1}{n}} \quad \forall\ n$$

$\Longrightarrow \lambda^*(G\setminus E)=0.$

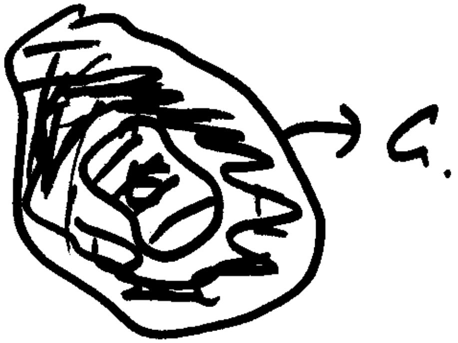
Hence $\underline{\text{(ii)} \Longrightarrow \text{(iii)}}$

<u>We show</u> (iii) $\Longrightarrow$ (i)

(iii) $\Longrightarrow$ $E\subseteq\mathbb{R}$, $\exists$ G_δ-set
$G\supseteq E$, $\lambda^*(G\setminus E)=0$

<u>Note</u>

$$E = \underset{\substack{\downarrow\\ \in\mathcal{B}_{\mathbb{R}}\\ \in\mathcal{L}}}{G}\cap(\underset{\substack{\downarrow\\ \in\mathcal{L}}}{\underline{G\setminus E}})^c$$



$\Longrightarrow$ $E\in\mathcal{L} \Longrightarrow$ (i)

(i) $E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$

$\Rightarrow \forall\ \varepsilon > 0, \exists$ an open set

$\underline{G_\varepsilon \supseteq E^c}$ and $\lambda^*(G_\varepsilon \setminus E^c) < \varepsilon$

$E \supseteq G_\varepsilon^c = C_\varepsilon$ is closed

and $E \setminus C_\varepsilon = E \cap (C_\varepsilon^c)$

$= E \cap G_\varepsilon = G_\varepsilon \setminus (E^c)$

$\underline{\text{No.6}}$ $\underline{\lambda^*(E \setminus C_\varepsilon)} = \lambda^*(G_\varepsilon \setminus E^c) \underline{< \varepsilon}$

$\Rightarrow$ (ii)

(ii) $\Longrightarrow$ (iii) $\forall\, \varepsilon > 0,\ \exists\ C_\varepsilon \subseteq E,$

C_ε closed $\lambda^*(E \setminus C_\varepsilon) < \varepsilon$

$\forall\ \varepsilon = 1/n,\ \exists\ C_n \subseteq E,$ C_n closed

such that $\lambda(E \setminus C_n) < 1/n$

Put $C := \bigcup_{n=1}^{\infty} C_n$, F_σ-set

$C \subseteq E$ with

$$\lambda^*(E \setminus C) \leq \lambda^*(E \setminus C_n) < \frac{1}{n}$$

$$\Longrightarrow \quad \lambda^*(E \setminus C) = 0.$$

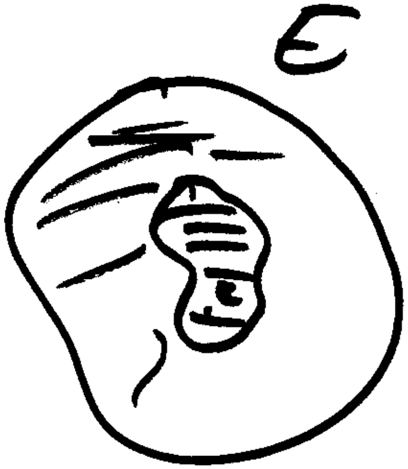
$\Longrightarrow$ (iii)

(iii) $\Longrightarrow$ (i)

$E \subseteq \mathbb{R}$

$\exists$ a F_σ-set $C \subseteq E$

$\lambda^*(E \setminus C) = 0$



N.b

$$E = \underset{\substack{\downarrow \\ \in \mathcal{B}_{\mathbb{R}} \\ \in \mathcal{L}}}{C} \cup \underset{\in \mathcal{L}}{(E \setminus C)}$$

$\Longrightarrow$ (i), $E \in \mathcal{L}$.