Lecture 12

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Measure and Integration

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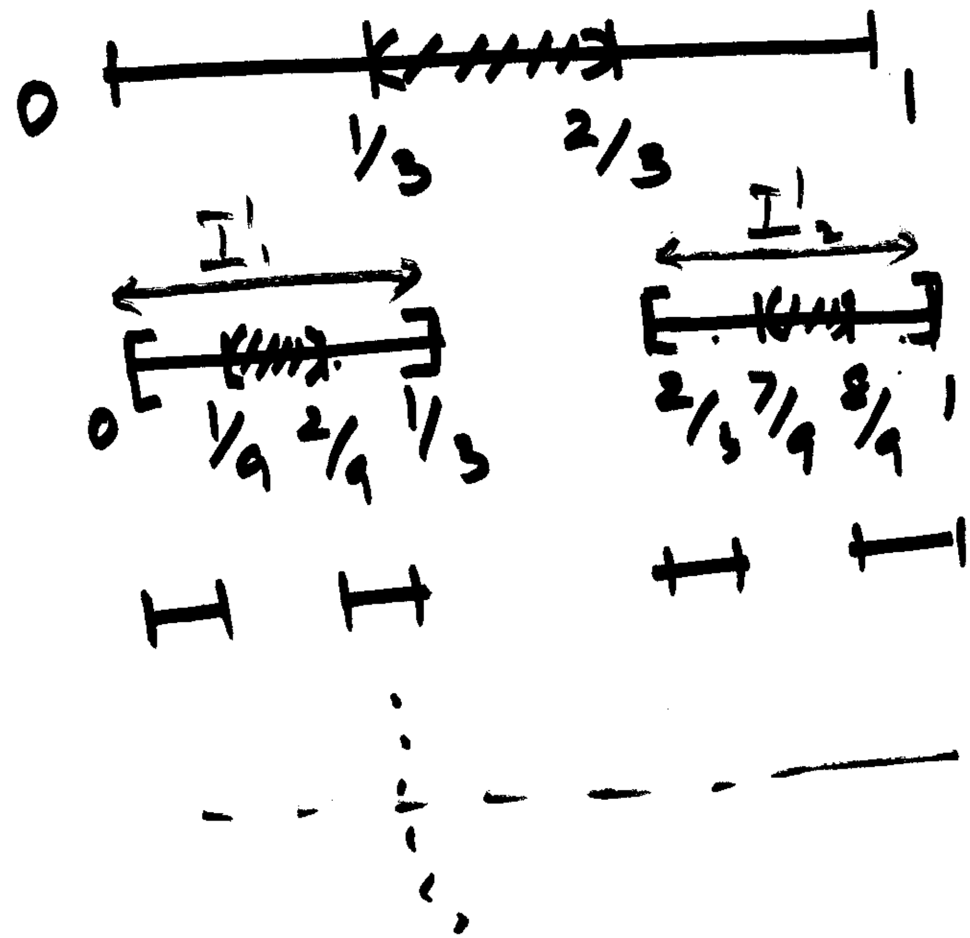
Cantor's Ternary set



$$A_0 = [0,1]$$

$$A_1 = I_1^1 \sqcup I_2^1$$

$$A_2 = \bigsqcup_{j=1}^{2^2} I_j^2$$

$$\vdots$$

$$A_n = \bigsqcup_{j=1}^{2^n} I_j^n$$

$$\vdots$$

Continue - - -

What is left is called Cantor's Ternary set C

1

$$x = .\, a_1 a_2 a_3 \cdots a_n \cdots$$

each $a_n = 0$ or 1

Construct a point y with ternary expansion

$$y = .\, b_1 b_2 \cdots b_n \cdots$$

where $\forall n \quad b_n = 2a_n$

<u>Note</u> $\quad y \in [0,1]$

$$\Rightarrow y \in C$$

$$x \in [0,1] \longmapsto y \in \cancel{[0,1]}\, C$$

It is one-one.

$$C = \bigcap_{n=1}^{\infty} A_n$$: Cantor's Ternary set.

Observations:

(1) The end points of the open intervals removed are in C: ($0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, 7/9, 8/9, \ldots$ will not be removed)

Thus $C \neq \emptyset$.

(2) In fact C is uncountable!!

$$[0,1] \xrightarrow[\text{one}]{\text{one}} C$$

Let $x \in [0,1]$, and consider its (!) binary expansion:

$$x_1 = \cdot a_1^1 a_2^1 \cdots a_n^1 \cdots$$

$$x_2 = \cdot a_1^2 a_2^2 \cdots a_n^2 \cdots$$

$x_1 \neq x_2 \implies \exists n_0$ such that

$$a_{n_0}^1 \neq a_{n_0}^2 .$$

$$\implies \begin{array}{ccc} 2a_{n_0}^1 & \neq & 2a_{n_0}^2 \\ \| & & \| \\ b_{n_0}^1 & & b_{n_0}^2 \end{array}$$

If $y_1 = \cdot b_1^1 b_2^1 \cdots b_n^1 \cdots$

$\quad y_2 = \cdot b_1^2 b_2^2 \cdots b_n^2 \cdots$

Then $y_1 \neq y_2$

(4)

Hence
$$\#(C) = \#[0,1]$$
$\Longrightarrow$ C is uncountable

Thus C is an uncountable set.

Note
$$C = \bigcap_{n=1}^{\infty} A_n$$
$$\Longrightarrow \forall n, \quad C \subseteq A_n = \bigsqcup_{j=1}^{2^n} I_j^n$$
where $\lambda(I_j^n) = \dfrac{1}{3^{2^n-1}}$
$$\sum_{j=1}^{2^n} \lambda(I_j^n) = \frac{2^n}{3^{2^n-1}} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\lambda^*(C) = 0$$

$\Rightarrow$ C is a λ^*-Null set

Hence $C \in \mathcal{L}$

<u>In fact</u> $E \subseteq C \Rightarrow \lambda^*(E) = 0$

$\Rightarrow E \in \mathcal{L}$.

Hence $\underline{\mathcal{P}(C)} \subseteq \mathcal{L} \subseteq \mathcal{P}(\underline{\mathbb{R}})$

$\Rightarrow$ '$\mathcal{L}$ has as many elements as $\mathcal{P}(\mathbb{R})$'

$$\#(\mathcal{L}) = \#\mathcal{P}(\mathbb{R}) = 2^c$$

Existence of non-measurable sets

$[0,1]$

For $x \sim y$ iff $x-y$ is a rational no.

Claims ① $x \sim y$ is an equivalence relation

$(x \sim x;\ x \sim y \Rightarrow y \sim x;\ x \sim y, y \sim z \Longrightarrow x \sim z)$

② $[0,1] = \bigsqcup_{\alpha \in A} E_\alpha$, E_α equivalence class

$E_\alpha \cap E_\beta = \phi$ for $\alpha \neq \beta$

xiom f hoic ③ From each E_α select $x_\alpha \in E_\alpha$ and form the set $E = \{x_\alpha : \alpha \in A\}$

$E \subset [0,1]$

Let rationals in $[-1,1]$ be written as $r_1, r_2, \dots, r_n, \dots$

Define $E_n = E + r_n$, $n \geq 1$

<u>Note</u>

$$E_n \subseteq [-1,2] \quad \forall\, n \geq 1$$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq [-1,2]$$

Also, $\underline{x \in [0,1]} \Rightarrow x \sim x_\alpha$ for some α

$\Rightarrow x - x_\alpha$ rational in $[-1,1]$

i.e. $r_n = (x - x_\alpha)$ $\cancel{\in E_n}$

$\Rightarrow x = x_\alpha + r_n \in E_n$

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} (E + r_n) \subseteq [-1, 2]$$

<u>Claim</u> $(E + r_n) \cap (E + r_m) = \phi$ for $n \neq m$

<u>If not</u> $x = x_\alpha + r_n = x_\beta + r_m$

$$\Rightarrow x \sim x_\alpha, \; x \sim x_\beta$$

$$\Rightarrow x_\alpha = x_\beta$$

$$\Rightarrow \alpha = \beta.$$

$$[0,1] \subseteq \bigsqcup_{n=1}^{\infty} (E + r_n) \subseteq [-1,2]$$

$\otimes$

Assume $\boxed{E \in \mathcal{L}}$

$\Rightarrow \lambda(E) = 0$

$$\lambda(E) = 0$$
$$\Rightarrow \lambda(E + r_n) = 0 \ \forall n$$
$$\Rightarrow \lambda\left(\bigsqcup_{n=1}^{\infty} (E + r_n)\right) = 0$$
$$\Rightarrow \lambda([0,1]) = 0$$

$\otimes$

$\Rightarrow \lambda(E) > 0$

$$\Rightarrow \lambda([-1,2]) = 3$$
$$\geq \lambda\left(\bigsqcup_{n=1}^{\infty} (E + r_n)\right)$$
$$= \sum_{n=1}^{\infty} \lambda(E + r_n)$$
$$= \sum_{n=1}^{\infty} \lambda(E)$$
$$= +\infty$$

$\otimes$