Lecture No. 11

Measure & Integration

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let $E_1, E_2 \in \mathcal{S}^*$ ①

To Show $E_1 \cup E_2$ is measurable.

E_1 mble $\Rightarrow \forall\ Y \subseteq X,\ \mu^*(Y) < +\infty$

$$\mu^*(Y) = \mu^*(Y \cap E_1) + \mu^*(Y \cap E_1^c)$$

Replace Y by $Y \cap (E_1 \cup E_2)$

then
$$(*)\quad \mu^*(Y \cap (E_1 \cup E_2)) = \mu^*(Y \cap E_1) + \underline{\mu^*(Y \cap E_1^c \cap E_2)}$$

Also E_2 is measurable. Thus

$$\mu^*(Y \cap E_1^c) = \mu^*\underline{(Y \cap E_1^c \cap E_2)} + \mu^*(Y \cap E_1^c \cap E_2^c)$$

Thus ③

$$\mu^*(Y\cap(E_1\cup E_2)) = \mu^*(Y\cap E_1) + \mu^*(Y\cap E_1^c) - \mu^*(Y\cap E_1^c\cap E_2^c)$$

$$\Rightarrow \mu^*(Y\cap(E_1\cup E_2)) + \mu^*(Y\cap E_1^c\cap E_2^c)$$
$$= \mu^*(Y\cap E_1) + \mu^*(Y\cap E_1^c)$$
$$= \mu^*(Y)$$

$\Rightarrow$ $E_1\cup E_2$ is measurable.

If E_1, E_2 are <u>disjoint</u> $E_1\cap E_2=\emptyset$

$\Rightarrow$ $\underline{E_2\subseteq E_1^c}$

In $\circledast$ put $Y = E_1 \cup E_2$ $(E_2 \subseteq E_1^c)$

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2).$$

Thus μ^* is finitely additive.

$A_n \in \mathcal{S}^*$, $n = 1, 2, \cdots$
pairwise disjoint, $A_n \cap A_m = \phi$ for $n \neq m$.

$A_1 \in \mathcal{S}^*$, $\forall Y \subseteq X$

$$\Rightarrow \mu^*(Y) = \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c)$$
$$= \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c \cap A_2) + \mu^*(Y \cap A_1^c \cap A_2^c)$$
$$= \mu^*(Y \cap A_1) + \mu^*(Y \cap A_2) + \mu^*(Y \cap A_1^c \cap A_2^c)$$

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$$= \sum_{i=1}^{n} \mu^*(Y \cap A_i) + \mu^*(Y \cap A_1^c \cap \cdots \cap A_n^c)$$

$$\mu^*(Y) = \sum_{i=1}^{n} \mu^*(Y \cap A_i) + \mu^*\left(Y \cap \left(\bigcup_{i=1}^{n} A_i\right)^c\right) \quad 5$$

$$\geq \sum_{i=1}^{n} \mu^*(Y \cap A_i) + \mu^*\left(Y \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

$$\rightarrow \quad \geq \sum_{i=1}^{\infty} \mu^*(Y \cap A_i) + \mu^*\left(Y \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

$$\geq \mu^*\left(Y \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + \mu^*\left(Y \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in S^*$$

In ⊛ put $Y = \bigcup_{i=1}^{\infty} A_i$.

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu^*(A_i) + 0$$

Also $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

$\Longrightarrow$ μ^* is countably additive on Σ^*.

Let $E \subseteq X$, $\mu^*(E) = 0$

Then $\forall\ Y \subseteq X$

$\mu^*(Y \cap E) = 0$ $\left(\because Y \cap E \subseteq E \atop \mu^* \text{ is monotone}\right)$

Thus $\mu^*(Y) \geq \mu^*(Y \cap E^c) + 0$
$= \mu^*(Y \cap E^c) + \mu^*(Y \cap E)$

$\Longrightarrow E \in S^*$

$$\begin{tikzcd}
\mu : \mathcal{A} \arrow[r, "\sigma\text{-finite}"] \arrow[d, bend left] \arrow[ddd, bend right] & {[0,\infty]} \\
\bar{\mu} : \mathcal{S}(\mathcal{A}) \arrow[r] & {[0,\infty]} \\
\bar{\mu} : \mathcal{S}^* \arrow[r, "\text{measure}"] & {[0,\infty]} \\
\mu^* : \mathcal{P}(X) \arrow[r, "\text{c.s.a}"] & {[0,+\infty]}
\end{tikzcd}$$

$\mathcal{S}(\mathcal{A}) \subseteq \mathcal{S}^* \subseteq \mathcal{P}(X)$

$\bar{\mu} = \mu^*|_{\mathcal{S}^*}$

μ^*

$$\underline{\mathcal{A} \subseteq \mathcal{S}(\mathcal{A})} \subseteq \mathcal{S}^* \subseteq \mathcal{P}(X)$$

$$\begin{tikzcd}
\mu \arrow[u] \arrow[rr, bend right] & \bar{\mu} & \mu^*
\end{tikzcd}$$

$E \subseteq X, \quad \mu^*(E) < \infty$

<u>Given $\varepsilon > 0$</u> $\exists\ F_\varepsilon \in \mathcal{A}$ such that

$$\mu^*(E \triangle F_\varepsilon) < \varepsilon.$$

$\mu^*(E) < +\infty$

$$\mu^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) \;\middle|\; \bigcup_{i=1}^{\infty} A_i \supseteq E,\ A_i \in \mathcal{A} \right\}$$

Given $\varepsilon > 0$, $\exists$ $A_i \in \mathcal{A}$, $E \subseteq \bigcup_{i=1}^{\infty} A_i$,

and $\mu^*(E) + \varepsilon/2 > \sum_{i=1}^{\infty} \mu(A_i)$ $\longleftarrow$

$\Longrightarrow$ $\sum_{i=1}^{\infty} \mu(A_i) < +\infty \Rightarrow \exists\, n_0$ s.t.

$$\sum_{i=n_0+1}^{\infty} \mu(A_i) < \varepsilon/2$$

Define $F_\varepsilon = \bigcup_{i=1}^{n_0} A_i \in \mathcal{A}$

$$E \setminus F_\varepsilon = E \setminus \left(\bigcup_{i=1}^{n_0} A_i \right)$$

$$\subseteq \left(\bigcup_{i=1}^{\infty} A_i \right) \setminus \left(\bigcup_{i=1}^{n_0} A_i \right)$$

$$\subseteq \bigcup_{i=n_0+1}^{\infty} A_i$$

$$\Longrightarrow \mu^*(E \setminus F_\varepsilon) \leq \mu^*\left(\bigcup_{i=n_0+1}^{\infty} A_i \right)$$

$$\leq \sum_{i=n_0+1}^{\infty} \mu^*(A_i) < \varepsilon/2 \quad \text{--- } (1)$$

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Also $\mu^*(F_\varepsilon \setminus E) = ?$

$$F_\varepsilon \setminus E = \left(\bigcup_{i=1}^{n_0} A_i\right) \setminus E$$

$$\subseteq \left(\bigcup_{i=1}^{\infty} A_i\right) \setminus E$$

$$\mu^*(F_\varepsilon \setminus E) \leq \sum_{i=1}^{\infty} \mu(A_i) - \mu^*(E)$$

$$< \varepsilon/2 \quad \text{②}$$

$$\mu^*(E \triangle F_\varepsilon) \leq \mu^*(E \setminus F_\varepsilon) + \mu^*(F_\varepsilon \setminus E)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$