Lecture 19

Measure & Integration

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$$\mu : \underset{\substack{\uparrow \\ \text{semi algebra}}}{\mathcal{C}} \longrightarrow [0,+\infty]$$

$$\tilde{\mu} : \underset{\substack{\uparrow \\ \text{Algebra generated} \\ \text{by } \mathcal{C}}}{\mathcal{F}(\mathcal{C})} \longrightarrow [0,+\infty]$$

$$\forall \; E \in \mathcal{F}(\mathcal{C}) \Longrightarrow E = \bigsqcup_{i=1}^{n} C_i, \; C_i \in \mathcal{C}$$

$$\underline{\tilde{\mu}(E)} = \tilde{\mu}\left(\bigsqcup_{i=1}^{n} C_i\right)$$

$$= \sum_{i=1}^{n} \tilde{\mu}(C_i)$$

$$= \sum_{i=1}^{n} \mu(C_i) \quad \checkmark$$

(ii) $\tilde{\mu}$ is well defined: ②

Suppose $E \in \mathcal{F}(\mathcal{C})$

$$E = \bigsqcup_{i=1}^{n} C_i = \bigsqcup_{j=1}^{m} D_j, \quad \begin{array}{l} C_i \in \mathcal{C} \\ D_j \in \mathcal{C} \end{array}$$

<u>To show</u> $\sum_{i=1}^{n} \mu(C_i) = \sum_{j=1}^{m} \mu(D_j)$?

<u>Note</u>

$$\bigsqcup_{i=1}^{n} C_i = \bigsqcup_{i=1}^{n} \left(C_i \cap \left(\bigsqcup_{j=1}^{m} D_j\right)\right)$$

$$= \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (C_i \cap D_j)$$

$$\bigsqcup_{j=1}^{m} D_j = \quad ||$$

$$\sum_{i=1}^{n} \mu(G_i) = \sum_{i=1}^{n} \left(\mu \left(\bigsqcup_{j=1}^{m} (G_i \cap \underline{D_j}) \right) \right) \quad (3)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \mu(G_i \cap D_j) \right) \text{ --- } (1)$$

$|||^{h}$

$$\sum_{j=1}^{m} \mu(D_j) = \sum_{j=1}^{m} \left(\mu \left(\bigsqcup_{i=1}^{n} D_j \cap G_i \right) \right)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(D_j \cap G_i) \text{ --- } (2)$$

(1) & (2)

$$\Rightarrow \sum_{i=1}^{n} \mu(G_i) = \sum_{j=1}^{m} \mu(D_j)$$

$\Rightarrow$ μ is well defined.

$\tilde{\mu}$ is finitely additive

let $E = \bigsqcup_{j=1}^{n} E_j$, $E_j \in \mathcal{F}(\mathcal{C})$
$E \in \mathcal{F}(\mathcal{C})$.

To show $\tilde{\mu}(E) = \sum_{j=1}^{n} \tilde{\mu}(E_j)$?

$$E_j \in \mathcal{F}(\mathcal{C}) \Rightarrow E_j = \bigsqcup_{k=1}^{n_j} E^j_k \ , E^j_k \in \mathcal{C}$$

$$\Rightarrow \underbrace{\bigsqcup_{j=1}^{n} E_j}_{\substack{\| \\ E}} = \bigsqcup_{j=1}^{n} \bigsqcup_{k=1}^{n_j} E^j_k .$$

$$\Rightarrow \tilde{\mu}(E) = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n_j} \mu(E^k_j) \Big)$$

$$\tilde{\mu}(E) = \sum_{j=1}^{n} \tilde{\mu}(E_j').$$

Hence $\tilde{\mu}$ is σ.a.

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Let $A, B \subseteq X$

$A \subseteq B$

To $\underline{\text{Show}}$ $\mu^*(A) \leq \mu^*(B)$

$$\mu^*(A) = \underline{\inf} \left\{ \sum_{i=1}^{\infty} \mu(E_i) \;\middle|\; \underline{A} \subseteq \bigsqcup_{i=1}^{\infty} E_i, \ E_i \in \mathcal{A} \right\}$$

$$\mu^*(B) = \underline{\inf} \left\{ \sum_{j=1}^{\infty} \mu(F_j) \;\middle|\; \underline{B} \subseteq \bigsqcup_{j=1}^{\infty} F_j, \ F_j \in \mathcal{A} \right\}$$

$A \subseteq B$, and $B \subseteq \bigsqcup_{j=1}^{\infty} F_j$

$$\Rightarrow \quad A \subseteq \bigsqcup_{j=1}^{\infty} F_j$$

$$\Rightarrow \quad \mu^*(A) \leq \mu^*(B).$$

μ^* is countably sub-additive? 7

To show: if $A \subseteq X$

and $A \subseteq \bigcup_{i=1}^{\infty} A_i$, $A_i \subseteq X$

Then $\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$? ✓

Note: if $\mu^*(A_i) = +\infty$ for some i,

then clearly

$$\mu^*(A) \leq +\infty \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

<u>Suppose</u> $\mu^*(A_i) < +\infty \ \forall i$ (8)

Let $\varepsilon > 0$ be arbitrary (fixed).

Then $\exists \ A_j^i$, $j = 1, 2, \ldots$ in $\mathcal{A}$ such that

$$A_i \subseteq \bigcup_{j=1}^{\infty} A_j^i$$

and $\mu^*(A_i) + \varepsilon/2^i > \sum_{j=1}^{\infty} \mu(A_j^i) \quad \forall i$

<u>Add</u> $\sum_{i=1}^{\infty} \mu^*(A_i) + \sum_i \varepsilon/2^i > \underline{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_j^i)}$

$\Rightarrow \quad \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon > \mu^*(A)$

$\therefore \quad \underline{A \subseteq \bigcup_i \bigcup_j A_j^i}, \ A_j^i \in C$

Letting $\varepsilon \to 0$, we have

$$\sum_{i=1}^{\infty} \mu^*(A_i) \geq \mu^*(A).$$

Hence μ^* is countably subadditive.

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) \;\middle|\; \underline{A \subseteq \bigcup A_i,\ A_i \in \mathcal{A}} \right\}$$

If $A \in \mathcal{A}$, then $A \subseteq A$

$\Longrightarrow \mu^*(A) \leq \mu(A)$ ✓

To Show $\mu(A) \leq \mu^*(A)$

<u>Case (1)</u> $\mu^*(A) = +\infty$, in that case

$\mu(A) \leq +\infty = \mu^*(A)$ ✓

<u>Case (2)</u> $\underline{\mu^*(A) < +\infty}$.

Let $\varepsilon > 0$ be arbitrary.

Then $\exists$ $A_j \in \mathcal{A}$ s.t

$$A \subseteq \bigcup_{j=1}^{\infty} A_j$$

and $$\mu^*(A) + \varepsilon > \sum_{j=1}^{\infty} \mu(A_j) \quad \text{——} \ (1)$$

Note $A \subseteq \bigcup A_j$,

μ measure $\Rightarrow$ μ is c. sub. add

$$\Rightarrow \mu(A) \le \sum_{j=1}^{\infty} \mu(A_j) \quad \text{——} \ (2)$$

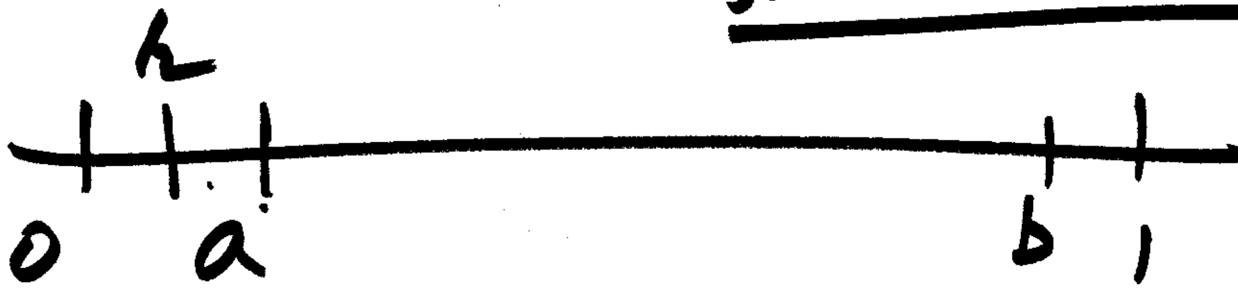
(1) + (2) $\Rightarrow$ $\mu^*(A) + \varepsilon > \mu(A)$

let $\varepsilon \to 0$, $\Rightarrow$ $\mu^*(A) \ge \mu(A)$.

$$E = \mathbb{Q} \cap (0,1)$$

$$E \subseteq \bigcup_{j=1}^{n} I_j' \qquad I_j \subset (0,1)$$

If possible let $\quad \sum_{j=1}^{n} \lambda(I_j') < 1 \quad$ (X)



$$I_j' = (a_j, b_j), \quad 1 \le j \le n$$

$$(a,b) \supseteq \bigcup_{j=1}^{n} I_j' \supseteq E$$

$$b - a \le \sum_{j=1}^{n} \lambda(I_j') < 1$$