Polynomial Interpolation:

Most important topic. Many results will be based on this topic.

Numerical Integration/differentiation

Solution of IVP/BVP

Root finding
Interpolation

\[ f: [a, b] \rightarrow \mathbb{R} \text{ continuous} \]

\[ x_0, x_1, \ldots, x_n : \text{distinct points in } [a, b] \]

To find a polynomial \( p_n \) of degree \( \leq n \) such that

\[ p_n(x_j) = f(x_j), \quad j = 0, 1, \ldots, n \]
$n = 0 : \quad p_n(x) = f(x_0) : \text{constant polynomial}$

$p_n(x_0) = f(x_0)$

$n = 1,$ fitting a straight line

$n = 2 : \quad \text{fitting a parabola}$
**General Case:**

$f: [a, b] \rightarrow \mathbb{R}$ continuous

$x_0, x_1, \ldots, x_n$: distinct points in $[a, b]$

Define

\[ l_i(x) = \frac{(x-x_0) \cdots (x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0) \cdots (x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \]

**Lagrange polynomial:** degree $n$

\[ l_i(x_i) = 1, \quad l_i(x_j) = 0 \text{ for } j \neq i \]
Interpolating Polynomial: Existence

\[ l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}, \quad l_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \]

\[ l_i: \text{ polynomial of degree } n \]

Let

\[ p_n(x) = \sum_{i=0}^{n} f(x_i) \cdot l_i(x). \]

Then \( p_n(x_j) = f(x_j), \quad j = 0, 1, \ldots, n \).
Let $p_n$ be a polynomial of degree $n$. Then by the Fundamental theorem of algebra, $p_n(z_1) = 0$ for some $z_1 \in \mathbb{C}$.

$p_n(x) = (x - z_1) q_{n-1}(x)$

$q_{n-1}$: polynomial of degree $n-1$

$p_n(x) = \alpha (x - z_1)^{m_1} \cdots (x - z_k)^{m_k}$,

$m_1 + \cdots + m_k = n$. \textit{Factorization Thm.}
A polynomial of degree \( n \) has exactly \( n \) zeroes, counted according to their multiplicities.

A non-zero polynomial of degree \( \leq n \) has at most \( n \) distinct zeroes.

If a polynomial of degree \( \leq n \) has more than \( n \) zeroes, then it is a zero polynomial.
Interpolating Polynomial: Uniqueness

Let \( f: [a, b] \rightarrow \mathbb{R} \) be continuous,
\( x_0, x_1, \ldots, x_n \) distinct points.
Let \( p_n \) and \( q_n \) be polynomials of degree \( \leq n \) such that
\[
p_n(x_j) = f(x_j) = q_n(x_j), \quad j = 0, 1, \ldots, n
\]
\[
\Rightarrow (p_n - q_n)(x_j) = 0 \Rightarrow p_n(x) = q_n(x)
\]
Theorem: Let \( f : [a, b] \to \mathbb{R} \) be continuous. \( x_0, x_1, \ldots, x_n \) : \( n+1 \) distinct points in \( [a, b] \). Then there exists a unique polynomial \( p_n \) of degree \( \leq n \) such that
\[
p_n(x_j) = f(x_j), \ j = 0, 1, \ldots, n
\]
Corollary: \( f = q_m \) : polynomial of degree \( m < n \),

\( a_0, x_1, \ldots, x_n \) : distinct points in \([a, b]\)

\( p_n \) : interpolating polynomial of degree \( \leq n \)

\[ p_n(x_j) = f(x_j), \ j = 0, 1, \ldots, n \]

\[ \Rightarrow p_n = q_m \]
Lagrange form of the interpolating polynomial

\[ p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x), \quad l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x-x_j}{x_i-x_j} \]

\[ p_n(x_j) = f(x_j), \quad j = 0, 1, \ldots, n \]

\[ p_{n+1} \text{ interpolates } f \text{ at } x_0, \ldots, x_n \text{ & } x_{n+1} \]

\[ p_{n+1}(x) = \sum_{i=0}^{n+1} f(x_i) \tilde{l}_i(x), \quad \tilde{l}_i(x) = \prod_{j=0}^{n+1} \frac{x-x_j}{x_i-x_j} \]

\[ \text{not recursive} \]
Divided Difference: Definition

\[ f : [a, b] \rightarrow \mathbb{R}, \]
\[ x_0, x_1, \ldots, x_n : n+1 \text{ distinct points in } [a, b] \]
\[ p_n : \text{unique interpolating polynomial}. \]

Define the divided difference

\[ f[x_0, x_1, \ldots, x_n] = \text{coefficient of } x^n \text{ in } p_n(x). \]
Properties of the divided difference

\[ f[x_0, x_1, \ldots, x_n] = \text{coefficient of } x^n \text{ in } p_n(x) \]

1. independent of the order of \( x_0, x_1, \ldots, x_n \).

2. If \( f \) is a polynomial of degree \( m < n \), then \( p_n(x) = f(x) \) and

\[ f[x_0, x_1, \ldots, x_n] = 0 \]
Recurrence Relation

Let $p_{n-1}$ and $q_{n-1}$ be polynomials of degree $\leq n-1$ such that

$$p_{n-1}(x_j) = f(x_j), \quad j = 0, 1, \ldots, n-1,$$

$$q_{n-1}(x_j) = f(x_j), \quad j = 1, 2, \ldots, n.$$ 

Consider

$$p_n(x) = \frac{(x-x_0)q_{n-1}(x) + (x_n-x)p_{n-1}(x)}{x_n-x_0}$$
Define \( p_n(x) = \frac{(x-x_0)q_{n-1}(x) + (x_n-x)p_{n-1}(x)}{(x_n-x_0)} \).
\[ p_{n-1}(x_j) = f(x_j), \; j = 0, 1, \ldots, n-1 \]
\[ q_{n-1}(x_j) = f(x_j), \; j = 1, 2, \ldots, n \]
\[ p_n(x) = \frac{(x-x_0)q_{n-1}(x) + (x_n-x)p_{n-1}(x)}{x_n-x_0} \]

For \( j = 1, \ldots, n-1 \),
\[ p_n(x_j) = \frac{(x_j-x_0)f(x_j) + (x_n-x_j)f(x_j)}{x_n-x_0} \]
\[ = f(x_j) \]
\[ p_n(x) = \frac{(x-x_0)q_{n-1}(x) + (x_n-x)p_{n-1}(x)}{x_n-x_0} \]

\( p_n \): interpolates \( f \) at \( x_0, x_1, \ldots, x_n \),

\( q_{n-1} \): interpolates \( f \) at \( x_1, \ldots, x_n \),

\( p_{n-1} \): interpolates \( f \) at \( x_0, \ldots, x_{n-1} \)

Coeff. of \( x^n \) in \( p_n(x) \)

\[ = \frac{\text{Coeff. of } x^{n-1} \text{ in } q_{n-1}(x) - p_{n-1}(x)}{x_n-x_0} \]
Coeff. of $x^n$ in $p_n(x)$

$= \frac{\text{Coeff. of } x^{n-1} \text{ in } q_{n-1}(x) - \text{Coeff. of } x^{n-1} \text{ in } p_{n-1}(x)}{x_n - x_0}$

$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$
Formula for \( f(x_0, x_1, \ldots, x_n) \)

\[ p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x), \quad l_i(x) = \frac{\prod_{j=0}^{n} (x-x_j)}{(x-x_i)} \]

\[ l_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \]

Define \( \omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n) \)

\[ \Rightarrow \omega'(x) = (x-x_1)\cdots(x-x_n) + (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots \]

\[ \Rightarrow \omega'(x_i) = \prod_{\substack{j=0 \atop j \neq i}}^{n} (x_i-x_j) \text{ and coeff. of } x^n \text{ in } l_i(x) = \frac{1}{\omega'(x_i)} \]
\[ p_n(x) = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)} l_i(x), \]

**Coefficient of** \( x^n \) **in** \( l_i(x) = \frac{1}{\omega'(x_i)} \)

Where \( \omega(x) = \prod_{j=0}^{n} (x - x_j) \)

\[ = \text{Coefficient of} \ x^n \text{ in} \ p_n(x) = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}. \]

\[ f [x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)} \]