Gaussian Integration
So far we have fixed the interpolation points $x_0, x_1, \ldots, x_n$ in $[a, b]$, constructed the interpolating polynomial

$$p_n(x) = \sum_{i=0}^{n} f(x_i) \cdot l_i(x),$$

where $l_i(x) = \frac{n}{i!} \frac{(x-x_j)}{(x_i-x_j)}$ : Lagrange polynomial

$$\text{and } \int_a^b f(x) \, dx \approx \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x) \, dx = \sum_{i=0}^{n} \omega_i f(x_i)$$
Starting Point:

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{i=0}^{n} w_i \, f(x_i) \]

Question: Is it possible to determine real numbers \( w_i \) and points \( x_i \) in \([a, b]\) so that the error is zero for polynomials of degree \( \leq 2n+1 \)?
Let $n = 0$. \[ \int_{a}^{b} f(x) \, dx \approx w_0 f(x_0) \]

To determine $w_0$ and $x_0 \in [a,b]$ such that error is zero for linear polynomials.

$(\Rightarrow) f(x) = 1, \quad b - a = w_0$

$f(x) = x, \quad \frac{b^2 - a^2}{2} = w_0 x_0 \Rightarrow x_0 = \frac{a + b}{2}$

\[ \int_{a}^{b} f(x) \, dx \approx (b - a) f\left(\frac{a + b}{2}\right) : \text{Midpoint Rule} \]
Let $n = 1$. \[ \int_a^b f(x) \, dx \approx w_0 f(x_0) + w_1 f(x_1) \]

To determine $w_0, w_1, x_0, x_1$ such that there is no error for polynomials of degree $\leq 3$.

1. $f(x) = 1$: $b - a = w_0 + w_1$

2. $f(x) = x$: $\frac{b^2 - a^2}{2} = w_0 x_0 + w_1 x_1$

3. $f(x) = x^2$: $\frac{b^3 - a^3}{3} = w_0 x_0^2 + w_1 x_1^2$ [Non-linear equations]

4. $f(x) = x^3$: $\frac{b^4 - a^4}{4} = w_0 x_0^3 + w_1 x_1^3$
Problem:

To determine $w_0, w_1, x_0, x_1$, such that

$$\int_a^b f(x) \, dx = w_0 f(x_0) + w_1 f(x_1)$$

is exact for polynomials of degree $\leq 3$. 

\[ f(x) = f(x_0) + \int_{x_0}^{x} f'(x) \, dx + f'[x_0, x_1, x] (x-x_0)(x-x_1) \]

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} \left[ f(x_0) + \int_{x_0}^{x} f'(x) \, dx \right] \, dx + \int_{a}^{b} f'[x_0, x_1, x] (x-x_0)(x-x_1) \, dx. \]

\(x_0, x_1\) are any points in \([a, b]\) \Rightarrow error = 0 for polynomials of degree \leq 1
error = \int_a^b f[x_0, x_1, x] (x-x_0) (x-x_1) \, dx.

Suppose that \( x_0 \) and \( x_1 \) are such that
\[
\int_a^b (x-x_0) (x-x_1) \, dx = 0.
\]

Then write
\[
f[x_0, x_1, x] = f[y_0, x_0, x_1] + f[y_0, x_0, x_1, x](x-y_0)
\]

error = \int_a^b f[y_0, x_0, x_1] \int_a^b w(x) \, dx + \int_a^b f[y_0, x_0, x_1, x] (x-y_0) w(x) \, dx
If \( \int_{a}^{b} (x-x_0)(x-x_1) \, dx = 0 \), then

\[
\text{error} = \int_{a}^{b} f[y_0, x_0, x_1, x](x-y_0)(x-x_0)(x-x_1) \, dx
\]

Note that if \( f \) is a quadratic polynomial, then \( f[y_0, x_0, x_1, x] = 0 \).

Hence \text{error} = 0.
If \( \int_{a}^{b} (x-x_0)(x-x_1) \, dx = 0 \), then

\[
\text{error} = \int_{a}^{b} f \left[ y_0, x_0, x_1, x \right] (x-y_0)(x-x_0)(x-x_1) \, dx
\]

If, in addition, \( \int_{a}^{b} (x-x_0)(x-x_1) \, dx = 0 \), then

\[
\text{error} = \int_{a}^{b} f \left[ y_1, y_0, x_0, x_1, x \right] (x-y_0)(x-y_1)(x-x_0)(x-x_1) \, dx
\]

= 0 if \( f \) is a cubic polynomial.
If $x_0, x_1 \in [a, b]$ are such that
\[ \int_a^b (x-x_0)(x-x_1) \, dx = 0, \int_a^b (x-x_0)(x-x_1)x \, dx = 0, \]
then
\[ \int_a^b f(x) \, dx = \int_a^b [f(x_0) + f \left[ x_0, x_1 \right] (x-x_0)] \, dx \]
\[ + \int_a^b f \left[ x_0, x_0, x_1, x_1, x \right] (x-x_0)^2 (x-x_1)^2 \, dx. \]

\[ = \omega_0 f(x_0) + \omega_1 f(x_0) + \frac{f^{(4)}(x)}{4!} \int_a^b (x-x_0)^2 (x-x_1)^2 \, dx \]
Thus, the problem of finding $\omega_0, \omega_1, x_0, x_1$ such that

$$\int_a^b f(x) \, dx = \omega_0 f(x_0) + \omega_1 f(x_1)$$

is exact for cubic polynomials reduces to finding $x_0, x_1$ such that

$$\int_a^b (x-x_0)(x-x_1) \, dx = 0, \quad \int_a^b (x-x_0)(x-x_1) \, dx = 0$$

and putting $\omega_0 = \int_a^b \frac{x-x_1}{x_0-x_1} \, dx$, $\omega_1 = \int_a^b \frac{x-x_0}{x_1-x_0} \, dx$. 
**Inner Product**

Let \( f, g \in C[a, b] \). Define

\[
< f, g > = \int_{a}^{b} f(x) g(x) \, dx
\]

Then

1) \( < f, f > \geq 0 \), \( < f, f > = 0 \iff f(x) \equiv 0 \).

2) \( < f, g > = < g, f > \)

3) \( < f_1 + f_2, g > = < f_1, g > + < f_2, g > \),

\( < \alpha f, g > = \alpha < f, g > \), \( \alpha \in \mathbb{R} \)
**Definition:** \( g_1, g_2, \ldots, g_n \in C[a, b] \) are said to be **orthogonal** if

\[
< g_i, g_j > = 0, \quad i \neq j.
\]

This is denoted by \( g_i \perp g_j, \quad i \neq j \)

**Example:** \( g_1(x) = 1 \), \( g_2(x) = x - \frac{a+b}{2} \).

\[
< g_1, g_2 > = \int_a^b (x - \frac{a+b}{2}) \, dx = 0
\]
**Definition:** $g_1, g_2, \ldots, g_n \in C[a, b]$ are said to be orthonormal if
\[
< g_i, g_j > = \begin{cases} 
1 & i=j \\
0 & i \neq j 
\end{cases}
\]

**Example:**
\[
g_1(x) = \frac{1}{\sqrt{b-a}}, \quad g_2(x) = \frac{x-a+b}{2} \frac{1}{(b-a)\sqrt{b-a}}
\]
Recall that we are trying to find $x_0, x_1$ in $[a, b]$ such that

1) $\int_a^b (x-x_0)(x-x_1) \, dx = 0$

2) $\int_a^b \frac{(x-x_0)(x-x_1)}{P_2(x)} \, dx = 0$.

Thus we want to find $p_2(x)$ such that $p_2 \perp f_0(x) = 1$ and $p_2 \perp f_1(x) = x$. 
**Gram-Schmidt Ortho-normalization**

Consider \( f_0(x) = 1 \), \( f_1(x) = x \), \( f_2(x) = x^2 \).

Define \( g_0(x) = \frac{f_0(x)}{\|f_0\|} : \text{Constant polynomial,} \)

\[ r_1(x) = f_1(x) - \langle f_1, g_0 \rangle g_0(x) \Rightarrow \langle r_1, g_0 \rangle = 0 \]

\[ g_1(x) = \frac{r_1(x)}{\|r_1\|} : \text{Linear polynomial,} \]

\[ r_2(x) = f_2(x) - \langle f_2, g_0 \rangle g_0(x) - \langle f_2, g_1 \rangle g_1(x) \]

\[ g_2(x) = \frac{r_2(x)}{\|r_2\|} : \text{Quadratic,} \quad \langle g_2, g_1 \rangle = \langle g_2, g_0 \rangle = 0 \]
Let $[a, b] = [-1, 1]$.

$$f_0(x) = 1 \implies \|f_0\| = \left( \int_{-1}^{1} dx \right)^{1/2} = \sqrt{2} \implies g_0(x) = \frac{1}{\sqrt{2}}$$

$$f_1(x) = x, \quad r_1(x) = x - \frac{1}{\sqrt{2}} \langle f_1, g_0 \rangle g_0 = x,$$

since $\langle f_1, g_0 \rangle = \int_{-1}^{1} \frac{x}{\sqrt{2}} dx = 0$.

$$\|r_1\| = \left( \int_{-1}^{1} x^2 dx \right)^{1/2} = \sqrt{\frac{2}{3}}, \quad g_1(x) = \sqrt{\frac{3}{2}} x$$
\[ g_0(x) = \frac{1}{\sqrt{2}}, \quad g_1(x) = \sqrt{\frac{3}{2}} x, \]

\[ \|g_0\|_2 = \|g_1\|_2 = 1, \quad \langle g_0, g_1 \rangle = 0. \]

\[ f_2(x) = x^2 \Rightarrow \langle f_2, g_0 \rangle = \int_{-\frac{\sqrt{2}}{2}}^{1} \frac{x^2}{\sqrt{2}} \, dx = \frac{\sqrt{2}}{3} \]

\[ \langle f_2, g_1 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}} x^3 \, dx = 0 \]

\[ r_2(x) = x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3} \]

\[ \langle r_2, g_0 \rangle = \langle r_2, g_1 \rangle = 0 \]
\[ g_0(x) = \frac{1}{\sqrt{2}}, \quad g_1(x) = \sqrt{\frac{3}{2}} \cdot x, \]

\[ r_2(x) = x^2 - \frac{1}{3}, \quad \langle r_2, g_0 \rangle = \langle r_2, g_1 \rangle = 0. \]

Thus

\[
\int \frac{1}{\sqrt{3}} (x + \frac{1}{\sqrt{3}})(x - \frac{1}{\sqrt{3}}) \, dx = 0, \quad \int \frac{1}{\sqrt{3}} (x + \frac{1}{\sqrt{3}})(x - \frac{1}{\sqrt{3}}) \, dx = 0
\]

\[ x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}. \]

Gauss points in \([-1, 1]\)