Module 2: General Topology
Lecture 3: More Preliminaries from general topology

Exercises:

1. Prove that any continuous function \( f : [-1, 1] \to [-1, 1] \) has a fixed point, that is to say, there exists a point \( x \in [-1, 1] \) such that \( f(x) = x \).

2. Prove that the unit interval \([0, 1]\) is connected. Is it true that if \( f : [0, 1] \to [0, 1] \) has connected graph then \( f \) is continuous? What if connectedness is replaced by path connectedness?

3. Suppose \( X \) is a locally compact, non-compact, connected Hausdorff space, is its one point compactification connected? What happens if \( X \) is already compact and Hausdorff?

4. Show that any connected metric space with more than one point must be uncountable. Hint: Use Tietze's extension theorem and the fact that the connected sets in the real line are intervals.

5. Show that the complement of a two dimensional linear subspace in \( \mathbb{R}^4 \) is connected. Hint: Denoting by \( V \) be the two dimensional vector space, show that \( \Sigma = \{ x/\|x\| \mid x \in \mathbb{R}^4 - V \} \) is connected using stereographic projection or otherwise.

6. How many connected components are there in the complement of the cone
\[
x_1^2 + x_2^2 + x_3^3 - x_4^2 = 0
\]
in \( \mathbb{R}^4 \)? Hint: The complement of this cone is filled up by families of hyperboloids. Examine if there is a connected set \( B \) meeting each member of a given family.

7. A map \( f : X \to Y \) is said to be a local homeomorphism if for \( x \in X \) there exist neighborhoods \( U \) of \( x \) and \( V \) of \( f(x) \) such that \( f \big|_U : U \to V \) is a homeomorphism. If \( f : X \to Y \) is a local homeomorphism and a proper map, then for each \( y \in Y \), \( f^{-1}(y) \) is a finite set. Show that the map \( f : \mathbb{C} - \{1, -1\} \to \mathbb{C} \) given by \( f(z) = z^3 - 3z \) is a local homeomorphism. Is it a proper map?