Module 4: Theory of covering space
Lecture 16: Lifting of paths and homotopies

Exercises:

1. Use the general results of this section to give an efficient and transparent proof that \( \pi_1(S^1, 1) = \mathbb{Z} \). First show that for any loop \( \gamma \) based at \( 1 \), the map \( \pi_1(S^1, 1) \to \mathbb{Z} \) given by \( [\gamma] \mapsto \tilde{\gamma}(1) \) is well defined by theorem 16.1, is a group homomorphism using uniqueness of lifts. Show that surjectivity follows from uniqueness of lifts and injectivity follows from theorem 16.1.

2. Let \( X \) be a topological spaces and \( a, b \in X \). A simple chain connecting \( a \) and \( b \) is a finite sequence \( U_1, U_2, \ldots, U_n \) of open sets such that \( a \in U_1, b \in U_n \) and for \( 1 \leq i < j \leq n, U_i \cap U_j \neq \emptyset \) implies \( j = i + 1 \).

Show that if \( X \) is a connected metric space and \( U \) is an open covering of \( X \) then any two points \( a, b \in X \) can be connected by a simple chain. This property is referred to as chain connectedness. Is \( \mathbb{Q} \) chain connected?

3. Use the above exercise to show that if \( X \) is a chain-connected space and \( p : \hat{X} \to X \) is a covering projection then for any pair of points \( x, y \in X \) the fibers \( p^{-1}(x) \) and \( p^{-1}(y) \) have the same cardinality. The point here is that \( X \) need not be path connected and the idea of using a path joining \( x \) and \( y \) as was done in the proof of theorem 14.4 is no longer available.

4. A toral knot is a group homomorphism \( \kappa : S^1 \to S^1 \times S^1 \) given by \( z \mapsto (z^m, z^n) \) where \( m, n \in \mathbb{N} \). Regarding the toral knot as a loop on the torus determine its lifts with respect to the covering projection \( \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \).

5. For the group homomorphism \( \kappa \) of the previous exercise describe the induced map \( \kappa_a \).