Problem set 5: Separable Extensions

Notation: Throughout these exercises, $F \subset K \subset L$ is a tower of fields. Assume that $\text{char } F = p > 0$ in the problems 4-10.

1. Let $\text{char } F = 0$ and $f(x) \in F[x]$ be a monic polynomial of positive degree. Let $d(x) = (f(x), f'(x))$. Show that $g(x) = f(x)/d(x)$ has same roots as $f(x)$ and $g(x)$ is separable.

2. Let $a \in L$ be separable over $F$. Show that $a$ is separable over $K$.

3. Show that an algebraic extension of a perfect field is perfect.

4. Let $f(x) = x^p^n - a \in F[x]$ where $n$ is a positive integer. Show that $f(x)$ is irreducible over $F$ if and only if $a \notin F^p$.

5. Let $([K : F], p) = 1$. Show that $K$ is a separable algebraic extension of $F$.

6. Show that $\cap_{i=0}^{\infty} F^{p^i}$ is the largest perfect subfield of $F$.

7. Let $f(x) \in F[x]$ be irreducible. Show that there exists an irreducible separable polynomial $g(x) \in F[x]$ and a positive integer $e$ such that $f(x) = g(x^{p^e})$. Show that all roots of $f(x)$ have same multiplicity $p^e$.

8. A polynomial $f(x) \in F[x]$ is called a $p$-polynomial if it is of the form $x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_m x$. Show that a monic polynomial of positive degree is a $p$-polynomial if and only if its roots form a finite subgroup of the additive group of a splitting field of $f(x)$ over $F$ and every root has same multiplicity $p^e$.

9. Let $t$ be an indeterminate. Show that the field extension $F(t)/F(t^p)$ is not separable.

10. Let $K = \mathbb{F}_p(t, w)$ be the rational function field in two indeterminates $t, w$ over $\mathbb{F}_p$. Let $L$ be the splitting field over $K$ of the polynomial $h(x) = f(x)g(x)$ where $f(x) = x^p - t$ and $g(x) = x^p - w$. Prove the following:
   (a) $f$ and $g$ are irreducible over $K$.
   (b) $[L : K] = p^2$.
   (c) $L/K$ is not separable.
   (d) $a^p \in K$ for all $a \in L$. 

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