(1) Let $F$ be a finite field with characteristic $p$. Prove that $|F| = p^n$ for some $n$.

(2) Using $f(x) = x^2 + x - 1$ and $g(x) = x^3 - x + 1$, construct finite fields containing 4, 8, 9, 27 elements. Write down multiplication tables for the fields with 4 and 9 elements and verify that the multiplicative groups of these fields are cyclic.

(3) Determine irreducible monic polynomials over $\mathbb{Q}$ for $1 + i$, $2 + \sqrt{3}$, and $1 + \sqrt[3]{2} + \sqrt[4]{4}$.

(4) Prove that $x^3 - 2$ and $x^3 - 3$ are irreducible over $\mathbb{Q}$. 

(5) Prove that $\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$. Find an irreducible polynomial of $\sqrt[2]{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

(6) Determine the degree $[\mathbb{Q}(\sqrt{3} + 2\sqrt{2}) : \mathbb{Q}]$.

(7) Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

(8) Let $K/F$ be an algebraic field extension and $R$ be a ring such that $F \subset R \subset K$. Show that $R$ is a field.

(9) Let $K/F$ be an extension of degree $n$.

(a) For any $a \in K$, prove that the map $\mu_a : K \to K$ defined by $\mu_a(x) = ax$ for all $x \in K$, is a linear transformation of the $F$-vector space $K$. Show that $K$ is isomorphic to a subfield of the ring $F^{n \times n}$ of $n \times n$ matrices with entries in $F$.

(b) Prove that $a$ is a root of the characteristic polynomial of $\mu_a$. Use this procedure to find monic polynomials satisfied by $\sqrt[2]{3}$ and $1 + \sqrt[2]{3} + \sqrt[4]{4}$.

(10) Let $K = \mathbb{Q}(\sqrt{d})$ for some square free integer $d$. Let $\alpha = a + b\sqrt{d} \in K$.

Use the basis $B = \{1, \sqrt{d}\}$ of $K$ over $F$ and find the matrix $M^B_B(\mu_\alpha)$ of $\mu_\alpha : K \to K$ with respect to $B$. Prove directly that the map $a + b\sqrt{d} \mapsto M^B_B(\mu_\alpha)$, is an isomorphism of fields.

(11) Prove that $-1$ is not a sum of squares in the field $\mathbb{Q}(\beta)$ where $\beta = \sqrt[2]{2} e^{2\pi i / 3}$.

(12) Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{Z}[x]$. Suppose that $f(0)$ and $f(1)$ are odd integers. Show that $f(x)$ has no integer roots.
(13) Let $R$ be an integral domain containing $\mathbb{C}$. Suppose that $R$ is a finite dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

(14) Let $k$ be a field and $x$ be an indeterminate. Let $y = x^3/(x + 1)$. Find the minimal polynomial of $x$ over $k(y)$.

(15) Find an algebraic extension $K$ of $\mathbb{Q}(x)$ such that the polynomial

$$f(y) = y^2 - x^3/(x^2 + 1) \in \mathbb{Q}(x)[y]$$

has a root in $K$. 