All pay auction:

Two player auction

$P_1, P_2$ — bidders

$b_1, b_2$ — bid of Player 2

bid of Player 1
Player with the highest bid wins. Both the player pay their bid irrespective of the outcome.

Let us assume that $v_1, v_2$ denote the valuations of $P_1, P_2$ respectively.
$V_1, V_2$ are independent

Distributed as uniform random variables in $[0, 1]$.

\[
\begin{align*}
b_1 &= \frac{1}{2} V_1^2 \quad \text{Nash} \\
b_2 &= \frac{1}{2} V_2^2 \quad \text{Equilibrium for the all pay auction}
\end{align*}
\]
Let us start by assuming that player $P_2$ is bidding $b_2 = \frac{1}{2} V_2^2$. Let player $P_1$ bid $b$.

$$\pi(b)$$ — expected payoff to player 1 as a function of bid $b$.

$$\pi(b) = \text{Pr}(\text{win}) \times (V_1 - b) + \text{Pr}(\text{loss}) \times (-b)$$
\[ P_1 \text{ wins if } b > b_2 = \frac{1}{2} v_2^2 \]
\[ \Rightarrow \frac{1}{2} v_2^2 \leq b \]
\[ \Rightarrow v_2 \leq \sqrt{2b} \]
\[ P_r(\text{win}) = P_r(v_2 \leq \sqrt{2b}) \]

\[ P_r(\{v_2 \leq \sqrt{2b}\}) \]
\[ = P_r(\{v_2 \in [0, \sqrt{2b}]\}) \]
\[ = \sqrt{2b} = P_r(\text{win}) \]
\[ P_r(\text{loss}) = 1 - P_r(\text{win}) = 1 - \sqrt{2b} . \]
\[ \Pi(b) = \sqrt{2}b \left( v_1 - b \right) + (1 - \sqrt{2}b)(-b) \]
\[ = \sqrt{2}b \cdot v_1 - b \sqrt{2}b - b + b \sqrt{2}b \]
\[ = \sqrt{2}b \cdot v_1 - b \]

\[ \Pi(b) = \sqrt{2}b \cdot v_1 - b \]

Expected payoff to player 1 as a function of bid \( b \).
\[
\frac{\partial \Pi(b)}{\partial b} = \sqrt{2} \cdot v_1 \cdot \frac{1}{2 \sqrt{b}} - 1 = 0
\]
\[
\sqrt{2} V_1 \frac{1}{2 \sqrt{b}} - 1 = 0
\]

\[
\Rightarrow \sqrt{b} = \frac{V_1}{\sqrt{2}}
\]

\[
b = \frac{1}{2} V_1^2
\]

Similarly, it can be shown that if \( b_1 = \frac{1}{2} V_1^2 \), then \( b = \frac{1}{2} V_2^2 \) is a best response bid for Player P2.
\( b_1 = \frac{1}{2} v_1^2 \) \( b_2 = \frac{1}{2} v_2^2 \) \( \) Nash Equilibrium of the Two player all-pay auction game.

Expected revenue of Two player all-pay auction:
\[ b_1 = \frac{1}{2} V_1^2 \]
\[ b_2 = \frac{1}{2} V_2^2 \]

Revenue: \[ b_1 + b_2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \]

Expected Revenue
\[
= \frac{1}{2} \mathbb{E} V_1^2 + \frac{1}{2} \mathbb{E} V_2^2 + \frac{1}{2} \mathbb{E} V_1^2 \mathbb{E} V_2^2
\]
\[
= \frac{1}{2} \int_0^1 V_1^2 f_{V_1}(V_1) dV_1 + \frac{1}{2} \int_0^1 V_2^2 f_{V_2}(V_2) dV_2
\]
\[
= \frac{1}{2} \int_0^1 V_1^2 dV_1 + \frac{1}{2} \int_0^1 V_2^2 dV_2
\]
\[ \frac{1}{2} \int_0^1 v_1^2 \, dv_1 + \frac{1}{2} \int_0^1 v_2^2 \, dv_2 \]

\[ = \frac{1}{2} \left. \frac{v_1^3}{3} \right|_0^1 + \frac{1}{2} \left. \frac{v_2^3}{3} \right|_0^1 \]

\[ = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} \]

\text{Equivalence principle.}