Sealed Bid First Price auction:

Consider a two-player auction $P_1, P_2$.

Two players participating in the auction.
$P_1, P_2$ submit their individual bids $b_1, b_2$ respectively for the object being auctioned.

These bids are `sealed'. Therefore, each player does NOT know the bid of the other player.

$P_1$ does NOT know $b_2$ of Player 2
$P_2$ does NOT know $b_1$ of Player 1
Player with highest bid wins the auction and pays an amount equal to his bid to get the object being auctioned.

If $b_1 \geq b_2$—then $P_1$ wins the auction and pays his bid $b_1$ to get the object. Player $2$ who has lost the auction does NOT pay anything.
on the other hand, if,
\[ b_2 > b_1 \]
then player 2 i.e. \( P_2 \) wins the auction and pays his bid \( b_2 \) to get the object.

First Price Auction:
Player with the highest bid wins the auction and pays an amount equal to his bid value.
Each player has a private valuation for the object. Valuation is what value the player assigns to the object.

\[ \text{Player 1} - V_1 \text{ valuations} \]
\[ \text{Player 2} - V_2 \]

\( V_1, V_2 \) are the valuations of player 1, player 2 respectively for the object being auctioned.
These valuations are private:

- Player 1 does NOT know the valuation $V_2$ of Player 2.
- Player 2 does NOT know the valuation $V_1$ of Player 1.

These private valuations are distributed uniformly in the interval $[0, 1]$. 

\[
\begin{align*}
F_{V_1}(v_1) & \\
f_{V_2}(v_2) & \\
0 & 1
\end{align*}
\]
This game is Bayesian in nature since there is uncertainty regarding the valuation of the other player.

Wish to analyze this game, to find the Nash equilibrium bidding strategy of each player.
We will demonstrate, that
the bidding strategy

\[ b_1 = \frac{1}{2} v_1 \]
\[ b_2 = \frac{1}{2} v_2 \]

Therefore, each player bidding
half his valuation is
the Nash equilibrium
bidding strategy.
we wish to demonstrate that $b_1 = \frac{1}{2} v_1$ and $b_2 = \frac{1}{2} v_2$ are best responses to each other.

let us assume that player 2 is bidding $b_2 = \frac{1}{2} v_2$. What is the best response bid $b$ of player 1?
\( \Pi(b) \) — denotes payoff to player 1 as a function of \( b \).

If player 1 wins the auction, i.e. \( b \geq b_2 \), then his payoff is:
\[ V_1 - b \]

Net payoff = \( V_1 - b \)

valuation \( \text{bid paid on winning the auction} \)
If player 1 loses the auction then his payoff is zero because he does not pay anything, neither does he get the object.

Average payoff to player 1 is given as:

\[
Pr(\text{win}) X (V_1 - b) + Pr(\text{loss}) X 0 = Pr(\text{win}) X (V_1 - b).
\]
\( T(b) = \text{Pr}(\text{win}) \times (V_1 - b) \)

Payoff to player 1 as a function of bid \( b \).

What is \( \text{Pr}(\text{win}) \) i.e. the probability of winning the auction?

To win \( b \geq b_2 = \frac{1}{2} V_2 \)

For player 1 to win \( b \geq \frac{1}{2} V_2 \Rightarrow V_2 \leq 2b \)
Since $V_2$ is distributed uniformly in $[0,1]$, we must have $V_2$ in $[0,2b]$.

Probability $V_2$ lies in $[0,2b]$

$$\int_0^{2b} F_{V_2}(v_2) \, dv_2$$

$$= \int_0^{2b} 1 \, dv_2 = V_2 \bigg|_0^{2b}$$

$$= 2b.$$
\[ P_r \text{ (win) for player 1 is } 2b \]

Therefore \( \pi(b) \) is,

\[
\pi(b) = 2b \times (V_1 - b) \\
= 2bV_1 - 2b^2
\]

\[
\pi(b) = 2bV_1 - 2b^2
\]

\[
\frac{\partial \pi(b)}{\partial b} = 2V_1 - 4b = 0
\]

\[
\boxed{b^* = \frac{1}{2} V_1}
\]
If $b_2 = \frac{1}{2} V_2$, then the bid $b_1 = \frac{1}{2} V_1$ is the best response of player 1.

Using a similar procedure it can be shown that if $b_1 = \frac{1}{2} V_1$, then $b_2 = \frac{1}{2} V_2$ is the best response of Player 2.
Therefore \( b_1 = \frac{1}{2} V_1 \), \( b_2 = \frac{1}{2} V_2 \) are best responses to each other.

\[
\begin{array}{c|c}
\hline
b_1 & \frac{1}{2} V_1 \\
\hline
b_2 & \frac{1}{2} V_2 \\
\end{array}
\]

Nash equilibrium of scaled bid first-price auction.