4.6 The Q R decomposition:

Let A be an nxn real nonsingular matrix. A matrix Q is said to be an orthogonal matrix if $QQ^T=I=Q^TQ$. We shall see that given the matrix we can find an orthogonal matrix Q and an upper triangular matrix R (with $r_{ii}>0$) such that A=QR called the QR decomposition of A. The Q and R are found as follows:

Let $a^{(1)}$; $a^{(2)}$; ……. , $a^{(n)}$ be the columns of A

$q^{(1)}$; $q^{(2)}$; ……… , $q^{(n)}$ be the columns of Q

$r^{(1)}$, $r^{(2)}$, ………., $r^{(n)}$ be the columns of R.

Note: Since Q is orthogonal we have

$$\left\|q^{(1)}\right\|_2 = 1 = \left\|q^{(2)}\right\|_2 = ... = \left\|q^{(n)}\right\|_2 ....(A)$$

$$\left(q^{(i)}, q^{(j)}\right) = 0 \qquad \text{if } i \neq j \quad ……………..\ (B).$$

and since R is upper triangular we have

$$r^{(i)} = \begin{pmatrix} r_{1i} \\ r_{2i} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix} ....(C)$$

Also the i^{th} column of QR is; $Qr^{(i)}$ and $\therefore$ i^{th} column of

$$QR = r_{1i}q^{(1)} + r_{2i}q^{(2)} + ...+ r_{ii}q^{(i)} ....(D)$$

We want A = QR.

Comparing 1^{st} column on both sides we get

$a^{(1)}$ = QR 's first column
$= Qr^{(1)}$
$= r_{11}q^{(1)}$ by (D)

$$\therefore \left\|a^{(1)}\right\|_2 = \left\|r_{11}q^{(1)}\right\|_2 = \left|r_{11}\right|\left\|q^{(1)}\right\|_2$$

$$= r_{11} \quad \because r_{11} > 0 \text{ and } \left\|q^1\right\|_2 = 1 by (A)$$

$$\therefore r_{11} = \left\|a^{(1)}\right\|_2 \text{ and } q^{(1)} = \frac{1}{r_{11}} a^{(1)} \text{..........} \text{.......}(E)$$

giving 1st columns of R and Q.

Next comparing second columns on both sides we get

$a^{(2)} = Qr^{(2)} = r_{12}\ q^{(1)} + r_{22}\ q^{(2)}$ ………. (*)

Therefore from (*) we get

$$\left(a^{(2)}, q^{(1)}\right) = r_{12}\left(q^{(1)}, q^{(1)}\right) + r_{22}\left(q^{(2)}, q^{(1)}\right)$$
$$= r_{12} \because \left(q^{(1)}, q^{(1)}\right) = \left\|q^{(1)}\right\|^2{}_2 = 1 by (A)$$

$$and \left(q^{(2)}, q^{(1)}\right) = 0 by (B)$$

$$\therefore r_{12} = \left(a^{(2)}, q^{(1)}\right) \text{...........}(F)$$

$\therefore$ (*) gives

$$r_{22}\, q^{(2)} = a^{(2)} - r_{12}\, q^{(1)}$$
$$and \ \therefore \left\|r_{22}\, q^{(2)}\right\|_2 = \left\|a^{(2)} - r_{12}\, q^{(1)}\right\|_2$$
$$\therefore r_{22} = \left\|a^{(2)} - r_{12}\, q^{(1)}\right\|_2 \text{..........} \text{..........} (G)$$
$$and$$
$$q^{(2)} = \frac{1}{r_{22}}\left[a^{(2)} - r_{12}\, q^{(1)}\right] \text{.........} \text{..........} \text{..}(H)$$

(F), (G), (H) give 2nd columns of Q and R. We can proceed having got the first i - 1 columns of Q and R we get ith columns of Q and R as follows:

$$r_{1i} = \left(a^{(i)}, q^{(1)}\right); r_{2i} = \left(a^{(1)}, q^{(2)}\right), \text{............}, r_{i-1i} = \left(a^{(i)}, q^{(i-1)}\right)$$

$$r_{ii} = \left\| a^{(i)} - r_{1i} q^{(1)} - r_{2i} a^{(2)} \ldots\ldots - r_{i-1i} q^{(i-1)} \right\|_2$$

$$q^{(i)} = \frac{i}{r_{ii}} \left[a^{(i)} - r_{1i} q^{(1)} - r_{2i} q^{(2)} \ldots\ldots\ldots - r_{i-1} q^{(i-1)} \right]$$

Example:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

1st column of Q and R

$$r_{11} = \left\| a^{(1)} \right\|_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$q^{(1)} = \frac{1}{r_{11}} a^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

2nd column of Q and R:

$$r_{12} = \left(a^{(2)}, q^{(1)} \right) = \left(\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$r_{22} = \left\| a^{(2)} - r_{12} q^{(1)} \right\|_2 = \left\| \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\|_2 = \sqrt{3}$$

$$q^{(2)} = \frac{1}{\sqrt{3}}\left[a^{(2)} - r_{12}q^{(1)}\right] = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

<u>3rd column of Q and R:</u>

$$r_{13} = \left(a^{(3)}, q^{(1)}\right) = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$r_{23} = \left(a^{(3)}, q^{(2)}\right) = \frac{1}{\sqrt{3}}$$

$$r_{33} = \left\| a^{(3)} - r_{13}q^{(1)} - r_{23}q^{(2)} \right\|$$

$$= \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\|_2$$

$$= \left\| \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} \right\|_2 = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{2/3} = \frac{\sqrt{2}}{\sqrt{3}}$$

and

$$q^{(3)} = \frac{1}{r_{33}}\left[a^{(3)} - r_{13}q^{(1)} - r_{23}q^{(2)}\right]$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \quad ; \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

and

$$QR = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = A$$

giving us QR decomposition of A.

The QR algorithm

Let A be any nonsingular nxn matrix.
Let $A = A_1 = Q_1 R_1$ be its QR decomposition.
Let $A_2 = R_1 Q_1$. Then find the QR decomposition of A_2 as $A_2 = Q_2 R_2$
Define $A_3 = R_2 Q_2$; find QR decomposition of A_3 as

$$A_3 = Q_3 R_3.$$

Keep repeating the process. Thus

$$A_1 = Q_1 R_1$$
$$A_2 = R_1 Q_1$$

and the i^{th} step is

$$A_i = R_{i-1} Q_{i-1}$$
$$A_i = Q_i R_i$$

Then A_i ‘ converges’ to an upper triangular matrix exhibiting the eigenvalues of A along the diagonal.