4.5 JACOBI ITERATION FOR FINDING EIGENVALUES OF A REAL SYMMETRIC MATRIX

Some Preliminaries:

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ be a real symmetric matrix.

Let $P = \begin{pmatrix} Cos\,\theta & -\sin\theta \\ \sin\theta & Cos\,\theta \end{pmatrix}$; (where we choose $|\theta| \le \pi/4$ for purposes of convergence of the scheme)

Note

$$P^t = \begin{pmatrix} Cos\,\theta & \sin\theta \\ -\sin\theta & Cos\,\theta \end{pmatrix} \text{ and } P^tP = PP^t = I$$

Thus P is an orthogonal matrix.

Now

$$A^1 = P^tAP = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a_{11}\cos\theta + a_{12}\sin\theta & -a_{11}\sin\theta + a_{12}\cos\theta \\ a_{12}\cos\theta + a_{22}\sin\theta & -a_{12}\sin\theta + a_{22}\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta & (-a_{11}+a_{22})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) \\ (-a_{11}+a_{22})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) & a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta \end{pmatrix}$$

Thus if we choose θ such that,

$$(-a_{11}+a_{22})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) = 0 \quad \ldots \text{(I)}$$

We get the entries in (1,2) position and (2,1) position of A^1 as zero.

(I) gives

$$\left(\frac{-a_{11}+a_{22}}{2}\right)\sin 2\theta + a_{12}(\cos 2\theta) = 0$$

$$\Rightarrow a_{12} \cos 2\theta = \frac{a_{11} - a_{22}}{2}\sin 2\theta$$

$$\Rightarrow \tan 2\theta = \frac{2a_{12}}{(a_{11}-a_{22})} = \frac{2a_{12}\operatorname{sgn}(a_{11}-a_{22})}{|a_{11}-a_{22}|}$$

$$= \frac{\alpha}{\beta}\text{, say } \quad ..... \text{(II)}$$

where $\alpha = 2a_{12}\operatorname{sgn}(a_{11}-a_{22})$ ..... (III)

$\beta = |a_{11} - a_{22}|$ ..... (IV)

$$\therefore \sec^2 2\theta = 1 + \tan^2 2\theta$$

$$= 1 + \frac{\alpha^2}{\beta^2} \quad \text{from (II)}$$

$$= \frac{\alpha^2+\beta^2}{\beta^2}$$

$$\therefore \cos^2 2\theta = \frac{\beta^2}{\alpha^2+\beta^2}$$

$$\therefore \cos 2\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \Rightarrow 2\cos^2\theta - 1 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$$

$$\Rightarrow \cos\theta = \sqrt{\frac{1}{2}\left[1+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right]} \qquad \text{. . . . . . . (V)}$$

and

$$2\sin\theta\cos\theta = \sin 2\theta = \sqrt{1-\cos^2 2\theta} = \sqrt{1-\frac{\beta^2}{\alpha^2+\beta^2}}$$

$$= \sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$$

$$\therefore \sin\theta = \frac{\alpha}{2\cos\theta\sqrt{\alpha^2+\beta^2}} \qquad \text{. . . . . .(VI)}$$

(V) and (VI) give sinθ, cosθ and if we choose

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ with these values of cosθ, sinθ, then

$P^tAP = A^1$ has (2,1) and (1,2) entries as zero.

We now generalize this idea.

Let $A = (a_{ij})$ be an nxn real symmetric matrix.

Let $1 \leq q < p < n$. (Instead of (1,2) position above choose (q, p) position)

Consider,

$$\alpha = 2a_{qp}\ \mathrm{sgn}(a_{qq} - a_{pp}) \qquad \text{. . . . . . . (A)}$$

$$\beta = \left| a_{qq} - a_{pp} \right| \qquad \ldots\ldots\ldots (B)$$

$$\cos\theta = \sqrt{\frac{1}{2}\left[1 + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right]} \qquad \ldots\ldots\ldots (C)$$

$$\sin\theta = \frac{1}{2\cos\theta}\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \qquad \ldots\ldots\ldots (D)$$

$$P = \begin{array}{cc} \begin{array}{cccccccc} & & & q & & p & & \\ & & & \downarrow & & \downarrow & & \end{array} & \\ \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\theta & & -\sin\theta & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & \sin\theta & & \cos\theta & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow q \\ \\ \\ \leftarrow p \\ \\ \\ \end{array} \end{array}$$

then $A^1 = P^t AP$ has the entries in (q, p) position and (p, q) position as zero.

In fact A^1 differs from A only in q^{th} row, p^{th} row and q^{th} column and p^{th} column and it can be shown that these new entries are

$$\left.\begin{aligned} a^1_{qi} &= a_{qi}\cos\theta + a_{pi}\sin\theta \\ a^1_{pi} &= -a_{qi}\sin\theta + a_{pi}\cos\theta \end{aligned}\right\} \quad i \neq q, p \ (q^{th} \text{ row } p^{th} \text{ row}) \ . . (E)$$

$$\left.\begin{aligned} a^1_{iq} &= a_{iq}\cos\theta + a_{ip}\sin\theta \\ a^1_{ip} &= -a_{iq}\sin\theta + a_{ip}\cos\theta \end{aligned}\right\} i \neq q,\ p\ (q^{th}\text{ column } p^{th}\text{ column}) \ . \ .(F)$$

$$\left.\begin{aligned} a^1_{qq} &= a_{qq}\cos^2\theta + 2a_{qp}\sin\theta\cos\theta + a_{pp}\sin^2\theta \\ a^1_{pp} &= a_{qq}\sin^2\theta - 2a_{qp}\sin\theta\cos\theta + a_{pp}\cos^2\theta \\ a^1_{qp} &= a^1_{pq} = 0. \end{aligned}\right\} \ . \ . \ . \ (G)$$

Now the Jacobi iteration is as follows.

Let A = (a_{ij}) be nxn real symmetric.

Find $1 \leq q < p \leq n$ such that $|a_{qp}|$ is largest among the absolute values of all the off diagonal entries in A.

For this q, p find P as above. Let $A^1 = P^t AP$. A^1 can be obtained as follows:
Except the p^{th} and the qth rows and the p^{th} and q^{th} columns other rows and columns of A^1 are the same as the corresponding rows and columns of A,

p^{th} row, q^{th} column, p^{th} column which are obtained from (E), (F), (G).

Now A^1 has 0 in (q, p), (p, q) position.

Replace A by A^1 and repeat the process. The process converges to a diagonal matrix the diagonal entries of which give the eigenvalues of A.

<u>Example:</u>

$$A = \begin{pmatrix} 7 & 3 & 2 & 1 \\ 3 & 9 & -2 & 4 \\ 2 & -2 & -4 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Entry with largest modulus is at (2, 4) position.

$\therefore$ q = 2, p = 4.

$$\alpha = 2\,\mathrm{sgn}\left(a_{qq} - a_{pp}\right).a_{qp} = 2\,\mathrm{sgn}\left(a_{22} - a_{44}\right).a_{24}$$

$$= (2)(1)(4) = 8\,.$$

$$\beta = \left|a_{qq} - a_{pp}\right| = \left|9 - 3\right| = 6$$

$$\therefore \alpha^2 + \beta^2 = 100\,; \qquad \sqrt{\alpha^2 + \beta^2} = 10$$

$$\therefore \cos\theta = \sqrt{\frac{1}{2}\left[\left(1 + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)\right]}$$

$$= \sqrt{\frac{1}{2}\left(1 + \frac{6}{10}\right)} = \sqrt{\frac{4}{5}} = \sqrt{0.8} = 0.89442$$

$$\sin\theta = \frac{1}{2\cos\theta}\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} = \frac{1}{2(0.89442)}\frac{8}{10}$$

$$= 0.44721$$

$$\therefore P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.89442 & 0 & -0.44721 \\ 0 & 0 & 1 & 0 \\ 0 & 0.44721 & 0 & 0.89442 \end{pmatrix}$$

$A^1 = P^TAP$ will have $a^1_{24} = a^1_{42} = 0$.

Other entries that are different from that of A are a^1_{21}, a^1_{22}, a^1_{23} ; a^1_{41}, a^1_{42}, a^1_{43}, a^1_{44} ; (of course by symmetric corresponding reflected entries also change).

We have,

$$a^1{}_{21} = a_{21}\cos\theta + a_{41}\sin\theta = 3.1305$$

$$a^1{}_{41} = -a_{21}\sin\theta + a_{41}\cos\theta = -0.44721$$

$$a^1{}_{23} = a_{23}\cos\theta + a_{43}\sin\theta = -0.89443$$

$$a^1{}_{43} = -a_{23}\sin\theta + a_{43}\cos\theta = 2.68328$$

$$a^1{}_{22} = a_{22}\cos^2\theta + 2a_{24}\sin\theta\cos\theta + a_{44}\sin^2\theta = 11$$

$$a^1{}_{44} = a_{22}\sin^2\theta - 2a_{24}\sin\theta\cos\theta + a_{44}\cos^2\theta = 1$$

$$\therefore A^1 = \begin{pmatrix} 7 & 3.1305 & 2 & -0.44721 \\ 3.1305 & 11 & -0.89443 & 0.0000 \\ 2 & -0.89443 & -4 & 2.68328 \\ -0.44721 & 0 & 2.68328 & 1.00000 \end{pmatrix}$$

Now we repeat the process with this matrix.

The largest absolute value is at (1, 2) position.

$\therefore$ q = 1, p = 2.

$$\beta = \left|a_{qq} - a_{pp}\right| = \left|a_{11} - a_{22}\right| = \left|7 - 11\right| = \left|-4\right| = 4$$

$$\alpha = 2a_{gp}\,\mathrm{sgn}\left(a_{qq} - a_{pp}\right) = 2(3.1305)(-1)$$

= - 6.2610.

$$\alpha^2 + \beta^2 = 55.200121$$
$$\sqrt{\alpha^2 + \beta^2} = 7.42968$$

$$\cos\theta = \sqrt{\frac{1}{2}\left[1 + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right]} = 0.87704 \;;$$

$$\sin\theta = \frac{1}{2\cos\theta}\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} = -0.48043$$

$\therefore$ The entries that change are

$$a^1{}_{12} = a^1{}_{21} = 0$$

$$a^1{}_{13} = a_{13}\cos\theta + a_{23}\sin\theta = 2.18378$$
$$a^1{}_{23} = -a_{13}\sin\theta + a_{23}\cos\theta = 0.17641$$

$$a^1{}_{14} = a_{14}\cos\theta + a_{24}\sin\theta = -0.39222$$
$$a^1{}_{24} = -a_{14}\sin\theta + a_{24}\cos\theta = -0.21485$$

$$a^1{}_{11} = a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta = 5.28516$$
$$a^1{}_{22} = a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta = 12.71484$$

and the new matrix is

$$\begin{pmatrix} 5.28516 & 0 & 2.18378 & -0.39222 \\ 0 & 12.71484 & 0.17641 & -0.21485 \\ 2.18378 & 0.17641 & -4 & 2.68328 \\ -0.39222 & -0.21485 & 2.68328 & 1 \end{pmatrix}$$

Now we repeat with q = 3, p = 4 and so on.

And at the 12th step we get the diagonal matrix

$$\begin{pmatrix} 5.78305 & 0 & 0 & 0 \\ 0 & 12.71986 & 0 & 0 \\ 0 & 0 & -5.60024 & 0 \\ 0 & 0 & 0 & 2.09733 \end{pmatrix}$$

giving eigenvalues of A as 5.78305, 12.71986, -5.60024, 2.09733.

Note: At each stage when we choose (q, p) position and apply the above transformation to get new matrix A^1 then sum of squares of off diagonal entries of A^1 will be less than that of A by $2a^2_{qp}$.