4.3 EIGENVALUES OF A REAL SYMMETRIC TRIDIAGONAL MATRIX

Let $$T = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & .... & 0 \\ b_1 & a_2 & b_2 & 0 & 0 & ..... & 0 \\ 0 & b_2 & a_3 & b_3 & 0 & ..... & 0 \\ ..... & ..... & ..... & ..... & ..... & ..... & ..... \\ 0 & ..... & ..... & 0 & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & ..... & ..... & 0 & b_{n-1} & a_n \end{pmatrix}$$

be a real symmetric tridiagonal matrix.

Let us find $P_n(\lambda) = \det[T - \lambda I]$

$$= \begin{vmatrix} a_1 - \lambda & b_1 & 0 & ..... & ..... & 0 \\ b_1 & a_2 - \lambda & b_2 & 0 & ..... & 0 \\ ..... & ..... & ..... & ..... & ..... & ..... \\ 0 & ..... & 0 & b_{n-2} & a_{n-1} - \lambda & b_{n-1} \\ 0 & ..... & ..... & 0 & b_{n-1} & a_n - \lambda \end{vmatrix}$$

The eigenvalues of T are precisely the roots of $P_n(\lambda) = 0$

(Without loss of generality we assume $b_i \neq 0$ for all i. For if $b_i = 0$ for some i then the above determinant reduces to two diagonal blocks of the same type and thus the problem reduces to that of the same type involving smaller sized matrices).

We define $P_i(\lambda)$ to be the i^{th} principal minor of the above determinant. We have

$$\left.\begin{aligned} &P_0(\lambda) = 1 \\ &P_1(\lambda) = a_1 - \lambda \\ &P_i(\lambda) = (a_i - \lambda)P_{i-1}(\lambda) - b^2{}_{i-1}P_{i-2}(\lambda) \end{aligned}\right\} \quad \text{........ (I)}$$

W are interested in finding the zeros of $P_n(\lambda)$. To do this we analyse the polynomials $P_0(\lambda), P_1(\lambda), \ldots\ldots, P_n(\lambda)$.

Let C be any real number. Compute $P_0(C), P_1(C), \ldots\ldots, P_n(C)$ (which can be calculated recursively by (I)). Let N (C) denote the agreements in sign between two consecutive in the above sequence of values, $P_0(C), P_1(C), \ldots\ldots, P_n(C)$. If for some i, $P_i(C) = 0$, we assign to it the the same sign as that of $P_{i-1}(C)$. Then we have

(F) There are exactly N (C) eigenvalues of T that are ≥ C.

<u>Example:</u>

If for an example we have an 8 x 8 matrix T (real symmetric tridiagonal) giving use to,

$$\left.\begin{aligned} &P_0(1) = 1 \\ &P_1(1) = 2 \end{aligned}\right\}$$
$$\left.\begin{aligned} &P_2(1) = -3 \\ &P_3(1) = -2 \end{aligned}\right\}$$
$$P_4(1) = 6$$
$$\left.\begin{aligned} &P_5(1) = -1 \\ &P_6(1) = 0 \end{aligned}\right\}$$
$$P_7(1) = 4$$
$$P_8(1) = -2$$

Here the consecutive pairs,

$$P_0(1), P_1(1)$$
$$P_2(1), P_3(1)$$
$$P_5(1), P_6(1)$$

agree in sign.

(Because since P_6 (1) = 0 we assign its sign to be the same as that of P_5 (1) .

Thus three pairs of sign agreements are achieved. Thus N (C) = 3; and there will be 3 eigenvalues of T greater than or equal to 1; and the remaining 5 eigen values are < 1.

It is this idea of result (F) that will be combined with (A), (B), (C), (D) and (E) of the previous section and clever repeated applications of (F) that will locate the eigenvalues of T. We now explain this by means of an example.

<u>Example 7:</u>

Let $$T = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ 0 & 4 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

Here we have

Absolute Row sum 1 = 3

Absolute Row sum 2 = 7

Absolute Row sum 3 = 7

Absolute Row sum 4 = 4

and therefore,

$$\|T\|_\infty = MARS = 7$$

(Note since T is symmetric we have MARS = MACS and therefore $\|T\|_1 = \|T\|_\infty = T$). Thus by our result (C), we have that the eigenvalues are all in the interval $-7 \le \lambda \le 7$

[——————————————————————————————]

-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7

Now the Gerschgorin (discs) intervals are as follows:

G_1 : Centre 1 radius : 2 $\therefore$ G_1 : [-1, 3]

G_2 : Centre -1 radius : 6 $\therefore$ G_2 : [-7, 5]

G_3 : Centre 2 radius : 5 $\therefore$ G_3 : [-3, 7]

G_4 : Centre 3 radius : 1 $\therefore$ G_4 : [2, 4]

We see that G_1, G_2, G_3 and G_4 all intersect to form one single connected region [-7, 7]. Thus by (E) there will be 4 eigenvalues in [-7, 7]. This gives therefore the same information as we obtained above using (C). Thus so far we know only that all the eigenvalues are in [-7, 7]. Now we shall see how we use (F) to locate the eigenvalues.

First of al let us see how many eigenvalues will be ≥ 0. Let C = 0. Find N (0) and we will get the number of eigenvalues ≥ 0 to be N (0).

Now

$$|T - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 & 0 \\ 2 & -1-\lambda & 4 & 0 \\ 0 & 4 & 2-\lambda & -1 \\ 0 & 0 & -1 & 3-\lambda \end{vmatrix}$$

$$P_0(\lambda) = 1 \qquad P_1(\lambda) = 1-\lambda \qquad \begin{aligned} P_2(\lambda) &= -(1+\lambda)P_1(\lambda) - 4P_0(\lambda) \\ P_3(\lambda) &= (2-\lambda)P_2(\lambda) - 16P_1(\lambda) \\ P_4(\lambda) &= (3-\lambda)P_3(\lambda) - P_2(\lambda) \end{aligned}$$

Now, we have,

$$P_0(0) = 1$$
$$P_1(0) = 1$$
$$P_2(0) = -5$$
$$P_3(0) = -26$$
$$P_4(0) = -73$$

We have

$$\left.\begin{array}{l} P_0(0), P_1(0) \\ P_2(0), P_3(0) \\ P_3(0), P_4(0) \end{array}\right\}$$ as three consecutive pairs having sign agreements.

$\therefore\ N(0) = 3$

$\therefore$ Three are <u>three</u> eigenvalues ≥ 0
and hence <u>one</u> eigenvalue < 0.

i.e. there are three eigenvalues in [0, 7]
and there is one eigenvalue in [-7, 0]

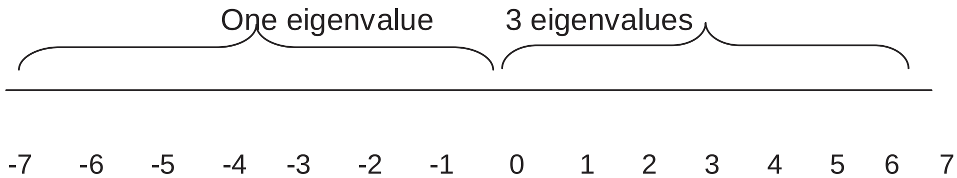
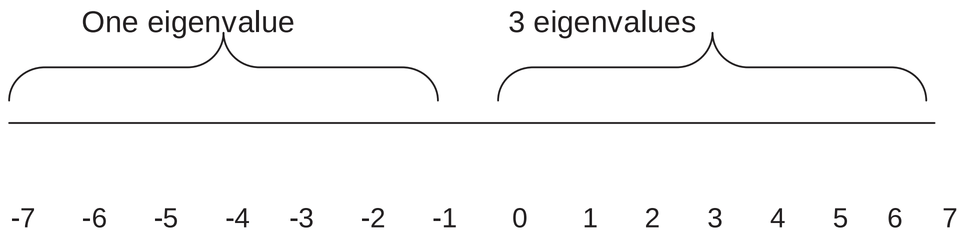


Fig.1

Let us take C = -1 and calculate N (C). We have

$$
\begin{aligned}
P_0(-1) &= 1\\
P_1(-1) &= 2\\
P_2(-1) &= -4\\
P_3(-1) &= -48\\
P_4(-1) &= -188
\end{aligned}
$$

Again we have N (-1) = 3. $\therefore$ There are three eigenvalues $\geq$ -1 Compare this with figure1. We get



(Fig.2)

Let us take the mid point of [-7, -1] in, which the negative eigenvalue lies.

So let C = -4.

P_0 (-4) = 1 Again there are three pairs of sign agreements.
P_1 (-4) = 5 $\therefore$ N (-4) = 3. $\therefore$ There are 3 eigenvalues $\geq$ -4.
P_2 (-4) = 11 Comparing with fig. 2 we get
P_3 (-4) = -14
P_4 (-4) = -109

that the negative eigenvalue is in [-7, -4] ………..(*)

Let us try mid pt. C = -5.5

We have

$P_0(-5.5) = 1$
$P_1(-5.5) = +6.5$ $\quad\therefore N(-5.5) = 4. \quad \therefore$ 4 eigenvalues ≥ -5.5.
$P_2(-5.5) = 25.25$ $\quad$ Combining this with (*) and fig. 2 we get
$P_3(-5.5) = 85.375$ $\quad$ that negative eigenvalue is in [-5.5 – 4].
$P_4(-5.5) = 683.4375$

We again take the mid pt. C and calculate N (C) and locate in which half of this interval does this negative eigenvalue lie and continue this bisection process until we trap this negative eigenvalue in as small an interval as necessary.

Now let us look at the eigenvalues ≥ 0. We have from fig. 2 three eigenvalues in [0, 7]. Now let us take C = 1

$P_0(1) = 1$
$P_1(1) = 0$ $\quad \therefore N(1) = 3$
$P_2(1) = -4$ $\quad \therefore$ all the eigenvalues are ≥ 1 ………….. (**)
$P_3(1) = -4$
$P_4(1) = -4$

C = 2

$P_0(2) = 1$
$P_1(2) = -1$ $\quad \therefore N(2) = 2 \quad \therefore$ There are two eigenvalues
$P_2(2) = -1$ $\quad \geq 2$. Combining this with (**) we get one
$P_3(2) = 16$ $\quad$ eigenvalue in [1, 2) and two in [2, 7].
$P_4(2) = 17$

<u>C = 3</u>

$P_0(3) = 1$ $\quad \therefore N(3) = 1 \quad \therefore$ one eigenvalue ≥ 3
$P_1(3) = -2$ $\quad$ Combining with above observation we get
$P_2(3) = 4$ $\quad$ one eigenvalue in [1, 2)
$P_3(3) = 28$ $\quad$ one eigenvalue in [2, 3)

P_4 (3) = - 4 one eigenvalue in [3, 7)

Let us locate the eigenvalue in [3, 7] a little better. Take C = mid point = 5

P_0 (5) = 1

P_1 (5) = - 4 $\therefore$ N (5) = 1

P_2 (5) = 20 $\therefore$ this eigenvalue is $\geq$ 5

P_3 (5) = 4

P_4 (5) = -28

$\therefore$ This eigenvalue is in [5, 7]

Let us take mid point C = 6

P_0 (6) = 1

P_1 (6) = - 5 $\therefore$ N (6) = 0

P_2 (6) = 31 $\therefore$ No eigenvalue $\geq$ 6

P_3 (6) = - 44 $\therefore$ the eigenvalue is in [5, 6)

P_4 (6) = 101

Thus combining all, we have,

one eigenvalue in [-5.5, -4)

one eigenvalue in [1, 2)

one eigenvalue in [2, 3)

one eigenvalue in [5, 6)

Each one of these locations can be further narrowed down by the bisection applied to each of these intervals.

We shall now discuss the method of obtaining a real symmetric tridiagonal T similar to a given real symmetric matrix A.