UNIT 4
EIGEN VALUE COMPUTATIONS

4.1 COMPUTATION OF EIGEN VALUES

In this section we shall discuss some standard methods for computing the eigenvalues of an nxn matrix. We shall also briefly discuss some methods for computing the eigenvectors corresponding to the eigenvalues.

We shall first discuss some results regarding the general location of the eigenvalues.

Let A = (a_{ij}) be an nxn matrix; and let λ_1, λ_2, ….., λ_n be its eigenvalues (including multiplicities). We defined

$$P = \|A\|_{xp} = \max\left\{|\lambda_1|, |\lambda_2|, \ldots\ldots, |\lambda_n|\right\}$$

Thus if we draw a circle of radius P about the origin in the complex plane, then all the eigenvalues of A will lie on or inside this closed disc. Thus we have

(A) If A is an nxn matrix then all the eigenvalues of A lie in the closed disc $\{\lambda : |\lambda| \leq P\}$ in the complex plane.

This result give us a disc inside which all the eigenvalues of A are located. However, to locate this circle we need P and to find P we need the eigenvalues. Thus this result is not practically useful. However, from a theoretical point of view, this suggests the possibility of locating all the eigenvalues in some disc. We shall now look for other discs which can be easily located and inside which the eigenvalues can all be trapped.

Let $\|A\|$ be any matrix norm. Then it can be shown that $P \leq \|A\|$. Thus if we draw a disc of radius $\|A\|$ and origin as center then this disc will be at least as big as the disc given in (A) above and hence will trap all the eigenvalues. Thus, the idea is to use a matrix norm, which is easy to compute. For example we can use $\|A\|_\infty$ or $\|A\|_1$ which are easily computed as Maximum Absolute Row Sums (MARS) or Maximum Absolutr Column Sums (MACS) respectively, that is,

$$\|A\|_\infty = \underbrace{\text{Max}}_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$$ and

$$\|A\|_1 = \underbrace{Max}_{1 \le j \le n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}$$

. Thus we have,

(B) If A is an nxn matrix then all its eigenvalues are trapped in the closed disc $\{\lambda : |\lambda| \le \|A\|_\infty\}$ or the disc $\{\lambda : |\lambda| \le \|A\|_1\}$. (The idea is to use $\|A\|_\infty$ if it is smaller than $\|A\|_1$, and $\|A\|_1$ if it is smaller than $\|A\|_\infty$).

COROLLORY

(C) If A is Hermitian, all its eigenvalues are real and hence all the eigenvalues lie in the intervals,

$$\{\lambda : -P \le \lambda \le P\} \quad \text{by (A)}$$

$$\left.\begin{aligned} &\{\lambda : -\|A\|_\infty \le \lambda \le \|A\|_\infty\} \\ &\{\lambda : -\|A\|_1 \le \lambda \le \|A\|_1\} \end{aligned}\right\} \text{ by (B).}$$

Example 1:

$$\text{Let} \quad A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 3 \\ 1 & 2 & 0 \end{pmatrix}$$

Here 'Row sums' are $P_1 = 4$ $\quad$ $P_2 = 6$ $\quad$ $P_3 = 3$

$$\therefore \|A\|_\infty = MARS = 6$$

Thus the eigenvalues are all in the disc ; $\{\lambda : |\lambda| \le 6\}$

The 'Column sums' are C1 = 3, C2 = 5, $C_3 = 5$.

$\therefore \|A\|_1 = MACS = 5$

$\therefore$ The eigenvalues are all in the disc, $\{\lambda : |\lambda| \leq 5\}$,

In this example $\|A\|_1 = 5 < \|A\|_\infty = 6$; and hence we use $\|A\|_1$ and get the smaller disc $\{\lambda : |\lambda| \leq 5\}$, inside which all eigenvalues are located.

The above results locate all the eigenvalues in one disc. The next set of results try to isolate these eigenvalues to some extent in smaller discs. These results are due to GERSCHGORIN.

Let A = (a_{ij}) be an nxn matrix.

The diagonal entries are

$$\xi_1 = a_{11};\ \xi_2 = a_{22}; \ldots..,\ \xi_n = a_{nn};$$

Now let P_i denote the sum of the absolute values of the off-diagonal entries of A in the i^{th} row.

$$P_i = |a_{i1}| + |a_{i2}| + \ldots.. + |a_{ii-1}| + |a_{ii+1}| + \ldots.. + |a_{in}|$$

Now consider the discs:

$$G_1 : \text{Centre } \xi_1; \text{radius } P_1 : \{\lambda : |\lambda - \xi_1| \leq P_1\}$$

$$G_2 : \text{Centre } \xi_{21}; \text{radius } P_2 : \{\lambda : |\lambda - \xi_2| \leq P_2\}$$

and in general

$$G_i : \text{Centre } \xi_i; \text{radius } P_i : \{\lambda : |\lambda - \xi_i| \leq P_i\}$$

Thus we get n discs $G_1, G_2, \ldots, G_n$. These are called the GERSCHGORIN DISCS of the matrix A.

The first result of Gerschgorin is the following:

(D) All eigenvalue of A must lie within the union of these Gerschgorin discs.

<u>Example 2:</u>

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 1 \\ 3 & 1 & -5 \end{pmatrix}$

The Gerschgorin discs are found as follows:

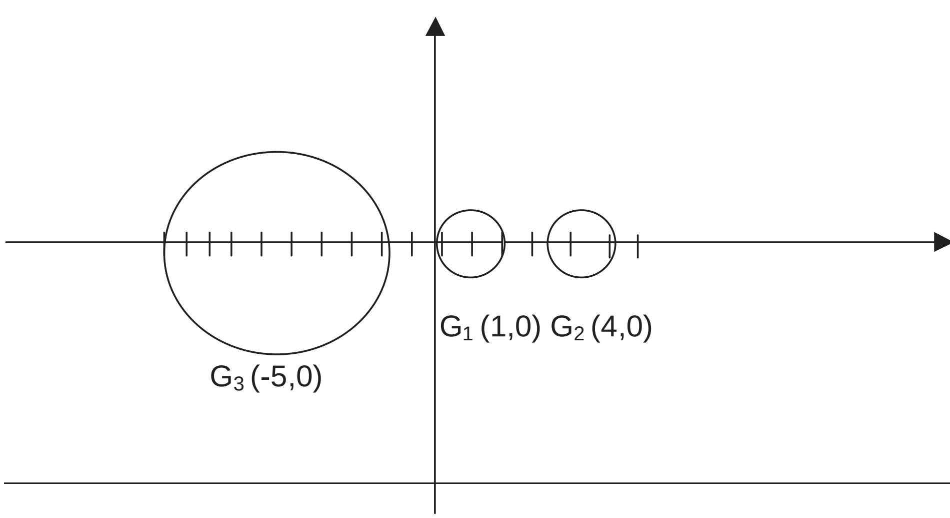
$\xi_1 = (1,0)$; $\xi_2 = (4,0)$; $\xi_3 = (-5,0)$

$P_1 = 1$; $P_2 = 1$; $P_3 = 4$

G_1 : Centre (1,0) radius 1

G_2 : Centre (4,0) radius 1

G_3 : Centre (-5,0) radius 4.



Thus every eigenvalue of A must lie in one of these three discs.

<u>Example 3:</u>

$$\text{Let} \quad A = \begin{pmatrix} 10 & 4 & 1 \\ 1 & 10 & 0.5 \\ 1.5 & -3 & 20 \end{pmatrix}$$

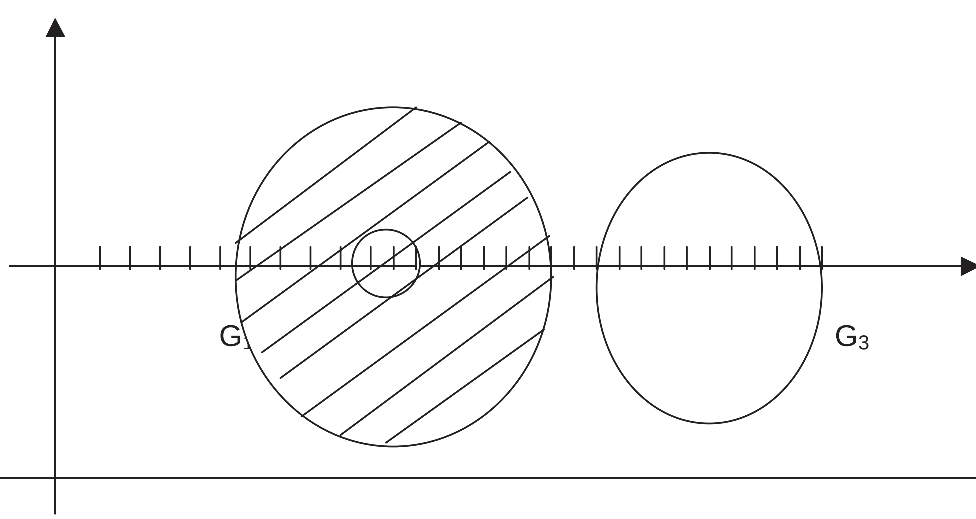
(It can be shown that the eigenvalues are exactly $\lambda_1 = 8$, $\lambda_2 = 12$, $\lambda_3 = 20$).

Now for this matrix we have,

$\xi_1 = (10,0)$ $\quad$ $\xi_2 = (10,0)$ $\quad$ $\xi_3 = 20$
$P_1 = 5$ $\quad$ $P_2 = 1.5$ $\quad$ $P_3 = 4.5$

Thus we have the three Gerschgorin discs

$$G_1 = \{\lambda : |\lambda - 10| \le 5\}$$
$$G_2 = \{\lambda : |\lambda - 10| \le 1.5\}$$
$$G_3 = \{\lambda : |\lambda - 20| \le 4.5\}$$



Thus all the eigenvalues of A are in these discs. But notice that our exact eigenvalues are 8,12 and 20. Thus no eigenvalue lies in G_2; and one eigenvalue lie in G_3 (namely 20) and two lie in G_1 (namely 8 and 12).

<u>Example 4:</u>

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

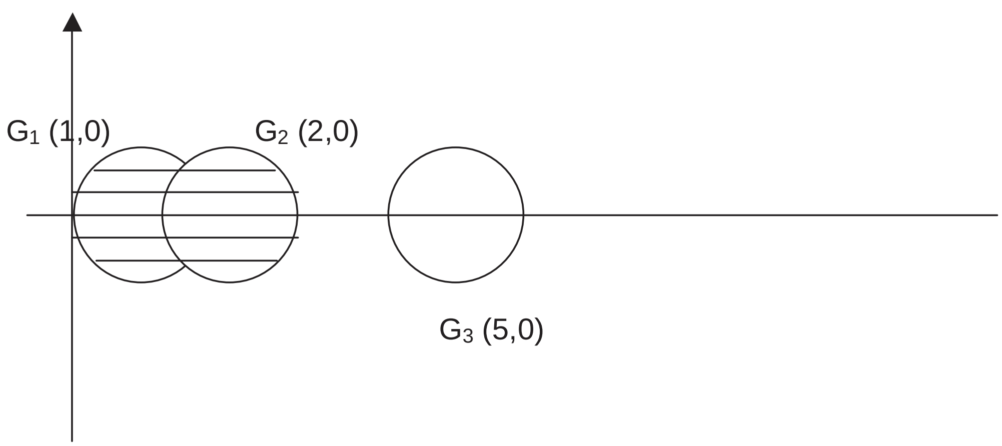
Now,

$\xi_1 = (1,0)$ $\xi_2 = (2,0)$ $\xi_3 = (5,0)$

$P_1 = 1$ $P_2 = 1$ $P_3 = 1$

The Gerschgorin discs are

$$G_1 = \{\lambda : |\lambda - 1| \le 1\}$$
$$G_2 = \{\lambda : |\lambda - 2| \le 1\}$$
$$G_3 = \{\lambda : |\lambda - 5| \le 1\}$$



Thus every eigenvalue of A must lie in the union of these three discs.

In example 2, all the Gerschgorin discs were isolated; and in examples 3 and 4 some discs intersected and others were isolated. The next Gerschgoin result is to identify the location of the eigenvalues in such cases.

(E) If m of the Gerschgorin discs intersect to form a common connected region and the remaining discs are isolated from this region then exactly m eigenvalues lie in this common region. In particular if Gerschgorin disc is isolated from all the rest then exactly one eigenvalue lies in this disc.

Thus in example 2 we have all three isolated discs and thus each disc will trap exactly one eigenvalue.

In example 3; $G_1$and G_2 intersected to form the connected (shaded) region and this is isolated from G_3. Thus the shaded region has two eigenvalues and G_3 has one eigenvalue.

In example 4, $G_1$and G_2 intersected to form a connected region (shaded portion) and this is isolated from G_3. Thus the shaded portion has two eigenvalues and G_3 has one eigenvalue.

<u>REMARK:</u>

In the case of Hermitian matrices, since all the eigenvalues are real, the Gerschgorin discs, $G_i = \left\{\lambda : \left|\lambda - a_{ii}\right| \le P_i\right\} = \left\{\lambda : \left|\lambda - \xi_i\right| \le P_i\right\}$ can be replaced by the Gerschgorin intervals,

$$G_i = \left\{\lambda : \left|\lambda - \xi_i\right| \le P_i\right\} = \left\{\lambda : \xi_i - P_i \le \lambda \le \xi_i + P_i\right\}$$

<u>Example 5:</u>

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 5 & 0 \\ 1 & 0 & -\frac{1}{2} \end{pmatrix}$

Note A is Hermitian. (In fact A is real symmetric)

Here; $\xi_1 = (1,0)$ $P_1 = 2$

$\xi_2 = (5,0)$ $P_2 = 1$

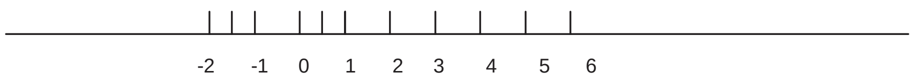
$\xi_3 = (-1/2,0)$ $P_3 = 1$

Thus the Gerschgorin intervals are

G_1 : $-1 \le \lambda \le 3$

G_2 : $4 \le \lambda \le 6$

G_3 : $-3/2 \le \lambda \le ½$



Note that G_1 and G_3 intersect and give a connected region, $-3/2 \le \lambda \le 3$; and this is isolated from G_2 : $4 \le \lambda \le 6$. Thus there will be two eigenvalues in $–3/2 \le \lambda \le 3$ and one eigenvalue in $4 \le \lambda \le 6$.

All the above results (A), (B), (C), (D), and (E) give us a location of the eigenvalues inside some discs and if the radii of these discs are small then the centers of these circles give us a good approximations of the eigenvalues. However if these discs are of large radius then we have to improve these approximations substantially. We shall now discuss this aspect of computing the eigenvalues more accurately. We shall first discuss the problem of computing the eigenvalues of a real symmetric matrix.

4.2 COMPUTATION OF THE EIGENVALUES OF A REAL SYMMETRIC MATRIX

We shall first discuss the method of reducing the given real symmetric matrix to a real symmetric tridiagonal matrix which is similar to the given matrix and then computing the eigenvalues of a real symmetric tridiagonal matrix. Thus the process of determining the eigenvalues of $A = (a_{ij})$, a real symmetric mtrix involves two steps:

STEP 1:

Find a real symmetric tridiagonal matrix T which is similar to A.

STEP 2:

Find the eigenvalues of T. (The eigenvalues of A will be same as those of T since A and T are similar).

We shall first discuss step 2.