3.4 GRAMM – SCHMIDT ORTHONORMALIZATION

Let U_1, U_2, …., U_k be k linearly independent vectors in $C^{n \text{ spanning a subspace } W}$. The Gramm – Schmidt process is the method to get an orthonormal set $\phi_1, \phi_2, \ldots, \phi_k$ such that the subspace ω spanned by U_1, ….., U_k is the same as the subspace spanned by $\phi_1, \ldots, \phi_k$ thus providing an orthonormal basis for W.

The process goes as follows:

Let $\psi_1 = U_1$;

$$\phi_1 = \frac{\psi_1}{\|\psi_1\|} = \frac{\psi_1}{\sqrt{(\psi_1, \psi_1)}} \text{ Note } \|\phi_1\| = 1$$

(We have used the symbol $\|x\|$ to denote the norm $\sqrt{(x, x)}$ of a vector x)

Next, let,

$$\psi_2 = U_2 - (U_2, \phi_1)\phi_1$$

Note that

$$\begin{aligned} &(\psi_2, \phi_1) \\ &= (U_2, \phi_1) - ((U_2, \phi_1)\phi_1, \phi_1) \\ &= (U_2, \phi_1) - (U_2, \phi_1)(\phi_1, \phi_1) \\ &= (U_2, \phi_1) - (U_2, \phi_1) \text{ (since } (\phi_1, \phi_1) = 1 \end{aligned}$$

Hence we get,

$$\psi_2 \perp \phi_1.$$

Let

$$\phi_2 = \frac{\psi_2}{\|\psi_2\|}; \qquad \text{clearly } \|\phi_2\| = 1, \|\phi_1\| = 1, (\phi_1, \phi_2) = 0$$

Also

x = α_1 U_1 + α_2 U_2 then

$$\Leftrightarrow x = \alpha_1 \psi_1 + \alpha_2 (\psi_2 + (U_2, \phi_1)\phi_1)$$

$$\Leftrightarrow x = \alpha_1 \|\psi_1\| \phi_1 + \alpha_2 [\|\psi_2\| \phi_2 + (U_2, \phi_1)\phi_1]$$

$\Leftrightarrow x = \beta_1\phi_1 + \beta_2\phi_2$, where

$$\beta_1 = \alpha_1\|\psi_1\| + \alpha_2(U_2, \phi_1)$$

$$\beta_2 = \alpha_2\|\psi_2\|$$

Thus $x \in$ subspace spanned by U_1, U_2

$\Leftrightarrow$ $x \in$ subspace spanned by ϕ_1, ϕ_2.

Thus ϕ_1, ϕ_2 is an orthonormal basis for the subspace $[U_1, U_2]$.

Having defined $\phi_1, \phi_2, \ldots, \phi_{i-1}$ we define ϕ_i as follows:

$$\psi_i = U_i - \sum_{p=1}^{i-1}(U_p, \phi_p)\phi_p \qquad \text{Clearly } (\psi_i, \phi_p) = 0 \quad 1 \le p \le i-1$$

and

$$\phi_i = \frac{\psi_i}{\|\psi_i\|}$$

Obviously $\|\phi_i\| = 1$ and $(\phi_i, \phi_j) = 0$ for $1 \le j \le i-1$

and $x \in$ the subspace spanned by $U_1, U_2, \ldots, U_i$ which we denote by $[U_1, U_2, \ldots, U_i]$

$\Leftrightarrow$ $x \in$ the subspace spanned by $\phi_1, \phi_2, \cdots, \phi_i$ which we denote by $[\phi_1, \ldots, \phi_i]$.

Thus $\phi_1, \phi_2, \ldots, \phi_i$ is an orthonormal basis for $[U_1, \ldots, U_i]$.

Thus at the k^{th} stage we get an orthonormal basis $\phi_1, \ldots, \phi_k$ for $[U_1, \ldots, U_k]$.

Example:

Let $U_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}; U_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}; U_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

be l.i. vectors in R^4. Let us find an orthonormal basis for the subspace ω spanned by U1, U2, U_3 using the Gramm – Schmidt process.

$$\psi_1 = U_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}; \qquad \phi_1 = \frac{\psi_1}{\sqrt{(\psi_1,\psi_1)}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$\therefore$

$$\phi_1 = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\end{pmatrix}$$

$$\psi_2 = U_2 - (U_2,\phi_1)\phi_1$$

$$= \begin{pmatrix}1\\1\\-1\\0\end{pmatrix} - \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\-1\\0\end{pmatrix} - \begin{pmatrix}1/3\\1/3\\\frac{1}{3}\\0\end{pmatrix}$$

$$= \begin{pmatrix}2/3\\2/3\\-4/3\\0\end{pmatrix} \qquad \text{and } \|\psi_2\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{16}{9}} = \frac{2\sqrt{6}}{3}$$

$$\therefore \phi_2 = \frac{\psi_2}{\|\psi_2\|} = \frac{3}{2\sqrt{6}}\begin{pmatrix} 2/3 \\ 2/3 \\ -4/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix}$$

Thus

$$\phi_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix}$$

Finally,

$$\psi_3 = U_3 - (U_3, \phi_1)\phi_1 - (U_3, \phi_2)\phi_2$$

$$= \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} - \left(\frac{6}{\sqrt{3}}\right)\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix} - \left(\frac{3}{\sqrt{6}}\right)\begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$$

$$\|\psi_3\| = \sqrt{1/4 + 1/4} = \sqrt{1/2} = 1/\sqrt{2}$$

$$\therefore \phi_3 = \frac{\psi_3}{\|\psi_3\|} = \sqrt{2}\begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

Thus the required orthonormal basis for W, the subspace spanned by U_1,U_2, U_3 is ϕ_1, ϕ_2, ϕ_3, where

$$\phi_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix}; \phi_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix}; \phi_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

Note that these ϕ_i are mutually orthogonal and have, each, 'length' one.

We now get back to Hermitian matrices. We had seen that the eigenvalues of a Hermitian matrix are all real; and that the eigenvectors corresponding to district eigenvalues are mutually orthogonal. We can further show the following: (We shall not give a proof here, but illustrate with an example).

Let A be any nxn Hermitian matrix. Let $C(\lambda) = (\lambda - \lambda_1)^{a_1}(\lambda - \lambda_2)^{a_2} .....(\lambda - \lambda_k)^{a_k}$ be its characteristic polynomial, where λ_1, λ_2, ….., λ_k are its distinct eigenvalues and a_1, ….., a_k are their algebraic multiplicities. If W_i is the characteristic subspace, (eigensubspace), corresponding to the eigenvalue λ_i ; that is,

$$\omega_i = \{x : Ax = \lambda_i x\}$$

then it can be shown that dim is $\omega_i = a_i$.

We choose any basis for ω_i and orthonormalize it by G-S process and get an orthonormal basis for ω_i. If we now take all these orthonormal - basis vectors for ω_1, . .,ω_k and write them as the columns of a matrix P then

P^*AP

will be a diagonal matrix.

<u>Example</u> :

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Notice

$A^* = \overline{A^1} = A^1 = A.$

Thus the matrix A is Hermitian.

<u>Characteristic</u> <u>Polynomial of A</u>:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 6 & 2 & -2 \\ 2 & \lambda - 3 & 1 \\ -2 & 1 & \lambda - 3 \end{vmatrix}$$

$$\xrightarrow{R_1 + 2R_2} \begin{vmatrix} \lambda - 2 & 2(\lambda - 2) & 0 \\ 2 & \lambda - 3 & 1 \\ -2 & 1 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 2) \begin{vmatrix} 1 & 2 & 0 \\ 2 & \lambda - 3 & 1 \\ -2 & 1 & \lambda - 3 \end{vmatrix}$$

$$\underset{R_3 + 2R_1}{\overset{R_2 - 2R_1}{\longrightarrow}} = (\lambda - 2) \begin{vmatrix} 1 & 2 & 0 \\ 0 & \lambda - 7 & 1 \\ 0 & 5 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 2)[(\lambda - 7)(\lambda - 3) - 5]$$

$$= (\lambda - 2)\left[\lambda^2 - 10\lambda + 16\right]$$

$$= (\lambda - 2)(\lambda - 2)(\lambda - 8)$$

$$= (\lambda - 2)^2 (\lambda - 8)$$

Thus

$$C(\lambda) = (\lambda - 2)^2 (\lambda - 8)$$

$\therefore$ $\lambda_1 = 2$ $a_1 = 2$

$\lambda_2 = 8$ $a_2 = 1$

The characteristic subspaces:

$$W_1 = \{x : Ax = 2x\}$$

$$= \{x : (A - 2I)x = \theta\}$$

i.e. We have to solve

$(A - 2I)\, x = \theta$

i.e. $\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow 2x_1 - x_2 + x_3 = 0$

$\Rightarrow x_3 = -2x_1 + x_2$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 + x_2 \end{pmatrix}; x_1, x_2 \text{ arbitrary}$$

$$\therefore \omega_1 = \left\{ x : x = \begin{pmatrix} \alpha \\ \beta \\ -2\alpha + \beta \end{pmatrix}; \alpha, \beta\, scalars \right\}$$

$\therefore$ A basis for W_1 is

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}; U_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

We now orthonormalize this:

$$\psi_1 = U_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \|\psi_1\| = \sqrt{5} \qquad \phi_1 = \frac{\psi_1}{\|\psi_1\|}$$

$$\therefore \phi_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\psi_2 = U_2 - (U_2, \phi_1)\phi_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left(-\frac{2}{\sqrt{5}}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ 0 \\ -\frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 \\ 1 \\ 1/5 \end{pmatrix}$$

$$\|\psi_2\| = \sqrt{\frac{4}{25}+1+\frac{1}{25}} = \sqrt{\frac{30}{25}} = \frac{\sqrt{30}}{5}$$

$$\therefore \phi_2 = \frac{\psi_2}{\|\psi_2\|} = \frac{5}{\sqrt{30}}\begin{pmatrix} 2/5 \\ 1 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ 1/\sqrt{30} \end{pmatrix}$$

$\therefore$ ϕ_1, ϕ_2 is an orthonormal basis for W_1.

$$W_2 = \{x : Ax = 8x\}$$

$$= \{x : (A-8I)x = \theta\}$$

So we have to solve

$(A-8I)\ x = \theta$ i.e.

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields $x_1 = -x_2 + x_3$ and therefore the general solution is

$$\begin{pmatrix} 2\gamma \\ \gamma \\ \gamma \end{pmatrix} = \gamma\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore Basis : U_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Orthonormalize: only one step:

$$\psi_3 = U_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\phi_3 = \frac{\psi_3}{\|\psi_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} \\ -\,1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$\therefore$ If $$P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & 5/\sqrt{30} & -\,1/\sqrt{6} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Then

P* = P^1 and

$$P^*AP = P^1AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix};$$

a diagonal matrix.