3. REVIEW OF PROPERTIES OF EIGENVALUES AND EIGENVECTORS

3.1 EIGENVALUES AND EIGENVECTORS

We shall now review some basic facts from matrix theory.

Let A be an nxn matrix. A scalar α is called an eigenvalue of A if there exists a nonzero nx1 vector x such that

$$Ax = \alpha x$$

Example:
Let

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

$$\alpha = -1$$

Consider

$$x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

We have

$$Ax = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$= (-1)x = \alpha x$$

Hence $\alpha = -1$ is such that there exists a nonzero vector x such that Ax = αx. Thus α is an eigenvalue of A.

Similarly, if we take α = 3, $x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ we find that Ax = αx. Thus, α = 3 is also an eigenvalue of A.

Let α be an eigenvalue of A. Then any nonzero x such that Ax = αx is called an eigenvector of A.

Let α be an eigenvalue of A. Let,

$$W_\alpha = \{x \in C^n : Ax = \alpha x\}$$

Then we have the following properties of W_α:

(i) W_α is nonempty, since the zero vector, (which we denote by θ), is in W_α, that is, $A\theta_n = \theta_n = \alpha\theta_n$.

(ii) $x, y \in W_\alpha \Rightarrow Ax = \alpha x, Ay = \alpha y$

$$\Rightarrow A(x+y) = \alpha(x+y)$$

$$\Rightarrow x + y \in W_\alpha$$

(iii) For any constant k, we have

$kAx = k\alpha x = \alpha(kx)$

$\Rightarrow A(kx) = \alpha(kx)$

$\Rightarrow kx \in W_\alpha$

Thus W_α is a subspace of C^n. This is called the characteristic subspace or the eigensubspace corresponding to the eigenvalue α.

Example: Consider the A in the example on page 81. We have seen that α = -1 is an eigenvalue of A. What is $W_{(-1)}$, the eigensubspace corresponding to –1?

We want to find all x such that

Ax = -x, that is,

(A+I)x = θ, that is,

we want to find all solutions of the homogeneous system Mx = θ ; where

$$M = A + I = \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix}$$

We now can use our row reduction to find the general solution of the system.

$$M \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{8}R_1} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $x_1 = \frac{1}{2}x_2 + \frac{1}{2}x_3$

Thus the general solution of (A+I) x = θ is

$$\begin{pmatrix} \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}x_2\begin{pmatrix}1\\2\\0\end{pmatrix} + \frac{1}{2}x_3\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$= A_1\begin{pmatrix}1\\2\\0\end{pmatrix} + A_2\begin{pmatrix}1\\0\\2\end{pmatrix}$$

where A_1 and A_2 are arbitrary constants.

Thus consists of all vectors of the form

$$A_1\begin{pmatrix}1\\2\\0\end{pmatrix} + A_2\begin{pmatrix}1\\0\\2\end{pmatrix}.$$

Note: The vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}$ form a basis for ω_1 and therefore

dim $W_{(-1)} = 2$.

What is $W_{(3)}$ the eigensubspace corresponding to the eigenvalue 3 for the above matrix?

We need to find all solutions of Ax = 3x,

i.e., Ax – 3x = θ

i.e., Nx = θ

where

$$N = A - 3I = \begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix}$$

Again we use row reduction

$$N \xrightarrow{R_1 - \frac{2}{3}R_3 \text{ and } R_1 - \frac{4}{3}R_1} \begin{pmatrix} -12 & 4 & 4 \\ 0 & -\frac{8}{3} & \frac{4}{3} \\ 0 & \frac{8}{3} & -\frac{4}{3} \end{pmatrix} \xrightarrow{R_4 + R_3} \begin{pmatrix} -12 & 4 & 4 \\ 0 & -\frac{8}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore 12x_1 = 4x_2 + 4x_3$

$\frac{8}{3}x_2 = \frac{4}{3}x_3 \qquad \therefore x_3 = 2x_2$

$\therefore 12x_1 = 4x_2 + 8x_2 = 12x_2$

$\therefore x_2 = x_1$

$\therefore x_2 = x_1 ; x_3 = 2x_2 = 2x_1$

$\therefore$ The general solution is

$$\begin{pmatrix} x_1 \\ x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Thus $W_{(3)}$ consists of all vectors of the form

$$\kappa \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Where κ is an arbitrary constant.

Note: The vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ forms a basis for $W_{(3)}$ and hence

dim. $W_{(3)} = 1$.

Now when can a scalar α be an eigenvalue of a matrix A of order n? We shall now investigate this question. Suppose α is an eigenvalue of A.

This $\Rightarrow$ There is a nonzero vector x such that Ax = αx.

$\Rightarrow (A - \alpha I)x = \theta$ and $x \neq \theta$

$\Rightarrow$ The system $(A - \alpha I)x = \theta$ has at least one nonzero solution.

$\Rightarrow$ nullity $(A - \alpha I) \geq 1$

$\Rightarrow$ rank $(A - \alpha I) < n$

$\Rightarrow$ $(A - \alpha I)$ is singular

$\Rightarrow$ det. $(A - \alpha I) = 0$

Thus, α is an eigenvalue of A $\Rightarrow$ det. $(A - \alpha I) = 0$.

Conversely, α is a scalar such that det. $(A - \alpha I) = 0$.

This $\Rightarrow$ $(A - \alpha I)$ is singular

$\Rightarrow$ rank $(A - \alpha I) < n$

$\Rightarrow$ nullity $(A - \alpha I) \geq 1$

$\Rightarrow$ The system $(A - \alpha I)x = \theta$ has nonzero solution.

$\Rightarrow$ α is an eigenvalue of A.

Thus, α is a scalar such that det. $(A - \alpha I) = 0 \Rightarrow \alpha$ is an eigenvalue.
Combining the two we get,
α is an eigenvalue of A
$\Leftrightarrow$ det. $(A - \alpha I) = 0$
$\Leftrightarrow$ det. $(\alpha I - A) = 0$

Now let $C(\lambda)$ = det. $(\lambda I - A)$

Thus we see that,
"The eigenvalues λ of a matrix A are precisely the roots of
$C(\lambda)$ = det. $(\lambda I - A)$".

We have,

$$C(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{vmatrix}$$

$$= \lambda^n - (a_{11} + \dots + a_{nn})\lambda^{n-1} + \dots + (-1)^n \det .A$$

Thus ; C(λ) is a polynomial of degree n. Note the 'leading' coefficient of C(λ) is one and hence C(λ) is a 'monic' polynomial of degree n. This is called CHARACTERISTIC POLYNOMIAL of A. The roots of the characteristic polynomial are the eigenvalues of A. The equation C(λ) = 0 is called the characteristic equation.

Sum of the roots of C(λ) = Sum of the eigenvalues of A

= a_{11} + ... + a_{nn} ,

and this is called the TRACE of A.

Product of the roots of C(λ) = Product of the eigenvalues of A

= det. A.

In our example in page 81 we have

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

$$\therefore C(\lambda) = \det .(\lambda I - A) = \begin{vmatrix} \lambda + 9 & -4 & -4 \\ 8 & \lambda - 3 & -4 \\ 16 & -8 & \lambda - 7 \end{vmatrix}$$

$$\xrightarrow{C_1 + C_2 + C_3} \begin{vmatrix} \lambda + 1 & -4 & -4 \\ \lambda + 1 & \lambda - 3 & -4 \\ \lambda + 1 & -8 & \lambda - 7 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} 1 & -4 & -4 \\ 1 & \lambda - 3 & -4 \\ 1 & -8 & \lambda - 7 \end{vmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} = (\lambda + 1)\begin{vmatrix} 1 & -4 & -4 \\ 0 & \lambda + 1 & 0 \\ 0 & -4 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda + 1)(\lambda + 1)(\lambda - 3)$$

$$= (\lambda + 1)^2 (\lambda - 3)$$

Thus the characteristic polynomial is

$$C(\lambda) = (\lambda + 1)^2 (\lambda - 3)$$

The eigenvalues are –1 (repeated twice) and 3.

Sum of eigenvalues = (-1) + (-1) + 3 = 1
= Trace A = Sum of diagonal entries.

Product of eigenvalues = (-1) (-1) (3) = 3 = det. A.

Thus, if A is an nxn matrix, we define the CHARACTERISTIC POLYNOMIAL as,

$$C(\lambda) = |\lambda I - A| \qquad . . . . (1)$$

and observe that this is a monic polynomial of degree n. When we factorize this as,

$$C(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \ldots\ldots (\lambda - \lambda_k)^{a_k} \qquad . . . . (2)$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ are the distinct roots; these distinct roots are the distinct eigenvalues of A and the multiplicities of these roots are called the algebraic multiplicities of these eigenvalues of A. Thus when $C(\lambda)$ is as in (2), the distinct eigenvalues are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ and the algebraic multiplicities of these eigenvalues are respectively, $a_1, a_2, \ldots, a_k$.

For the matrix in Example in page 81 we have found the characteristic polynomial on page 86 as

$$C(\lambda) = (\lambda + 1)^2 (\lambda - 3)$$

Thus the distinct eigenvalues of this matrix are $\lambda_1 = -1$; and $\lambda_2 = 3$ and their algebraic multiplicities are respectively $a_1 = 2$; $a_2 = 1$.

If λ_i is an eigenvalues of A, the eigen subspace corresponding to λ_i is W_{λ_i} and is defined as

$$W_{\lambda_i} = \{x \in C^n : Ax = \lambda_i x\}$$

The dimension of W_{λ_i} is called the <u>GEOMETRIC MULTIPLICITY</u> of the eigenvalue λ_i and is denoted by g_i.

Again for the matrix on page 81, we have found on pages 83 and 84 respectively that, dim $W_{(-1)} = 2$; and dim. $W_{(3)} = 1$. Thus the geometric multiplicities of the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ are respectively $g_1 = 2$; $g_2 = 1$. Notice that in this example it turns out that $a_1 = g_1 = 2$; and $a_2 = g_2 = 1$. In general this may not be so. It can be shown that for any matrix A having $C(\lambda)$ as in (2),

$$1 \leq g_i \leq a_i \ ; \ 1 \leq i \leq k \qquad \ldots\ldots\ldots\ldots(3)$$

i.e., for any eigenvalue λ_i of A, that is,
1 ≤ geometric multiplicity ≤ algebraic multiplicity for any eigenvalue.

We shall now study the properties of the eigenvalues and eigenvectors of a matrix. We shall start with a preliminary remark on Lagrange Interpolation polynomials :

Let $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_s$ be s distinct scalars, (i.e., $\alpha_i \neq \alpha_j$ if $i \neq j$). Consider,

$$p_i(\lambda) = \frac{(\lambda-\alpha_1)(\lambda-\alpha_2)\ldots(\lambda-\alpha_{i-1})(\lambda-\alpha_{i+1})\ldots(\lambda-\alpha_s)}{(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\ldots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\ldots(\alpha_i-\alpha_s)}$$

$$= \prod_{\substack{1\leq j\leq s \\ j\neq i}} \frac{(\lambda-\alpha_j)}{(\alpha_i-\alpha_j)} \qquad \text{for } i = 1,2,\ldots\ldots, s \qquad \ldots\ldots\ldots (4)$$

Then $p_i(\lambda)$ are all polynomials of degree s-1.

Further notice that $p_i(\alpha_1) = \ldots = p_i(\alpha_{i-1}) = p_i(\alpha_{i+1}) = \ldots = p_i(\alpha_s) = 0$

$$p_i(\alpha_i) = 1$$

Thus $p_i(\lambda)$are all polynomials of degree s-1 such that,

$$p_i(\alpha_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad \ldots\ldots\ldots\ldots (5)$$

We call these the Lagrange Interpolation polynomials. If $p(\lambda)$ is any polynomial of degree $\leq$ s-1 then it can be written as a linear combination of $p_1(\lambda), p_2(\lambda), \ldots, p_s(\lambda)$ as follows:

$$p(\lambda) = p(\alpha_1)p_1(\lambda) + p(\alpha_2)p_2(\lambda) + \cdots + p(\alpha_s)p_s(\lambda) \quad \ldots\ldots (6)$$

$$= \sum_{i=1}^{s} p(\alpha_i)p_i(\lambda)$$

With this preliminary, we now proceed to study the properties of the eigenvalues and eigenvectors of an nxn matrix A.

Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Let $\phi_1, \phi_2, \ldots, \phi_k$ be eigenvectors corresponding to these eigenvalues respectively ; i.e., ϕ_i are nonzero vectors such that

$$A\phi_i = \lambda_i\phi_i, \; i=1,2,\ldots,k \ldots\ldots\ldots\ldots (6)$$

From (6) it follows that

$$A^2\phi_i = A(A\phi_i) = A(\lambda_i\phi_i) = \lambda_i A\phi_i = \lambda^2{}_i\phi_i$$

$$A^3\phi_i = A(A^2\phi_i) = A(\lambda_i{}^2\phi_i) = \lambda_i{}^2{}_i A\phi_i = \lambda_i{}^3\ \phi_i$$

and by induction we get

$$A^m\phi_i = \lambda_i{}^m\ \phi_i \quad \text{for any integer } m \geq 0 \quad \ldots\ldots\ldots\ldots (7)$$

(We interpret A^0 as I).

Now,

$$p(\lambda) = a_0 + a_1\lambda + \ldots\ldots + a_s\lambda^s$$

be any polynomial. We define p(A) as the matrix,

$$p(A) = a_0 I + a_1 A + \ldots\ldots + a_s A^s$$

Now

$$\begin{aligned} p(A)\phi_i &= (a_0 I + a_1 A + \ldots\ldots + a_s A^s)\phi_i \\ &= a_0\phi_i + a_1 A\phi_i + \ldots\ldots + a_s A^s\phi_i \\ &= a_0\phi_i + a_1\lambda_i\phi_i + \ldots\ldots + a_s\lambda_i{}^s\ \phi_i \qquad \text{by (6)} \\ &= (a_0 + a_1\lambda_i + \ldots\ldots + a_s\lambda_i{}^s)\phi_i \\ &= p(\lambda_i)\phi_i. \end{aligned}$$

Thus,

If λ_i is any eigenvalue of A and ϕ_i is an eigenvector corresponding to λ_i ,then for any polynomial p(λ) we have $p(A)\phi_i = p(\lambda_i)\phi_i$. PROPERTY I

Now we shall prove that the eigenvectors, $\phi_1, \phi_2, \ldots, \phi_k$ corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A, are linearly independent .

In order to establish this linear independence, we must show that

$$C_1\phi_1 + C_2\phi_2 + \ldots + C_K\phi_K = \theta_n \Rightarrow C_1 = C_2 = \ldots = C_K = 0 \quad \ldots (8)$$

Now if in (4) & (5) we take s = k ; $\alpha_i = \lambda_i$, (I=1,2,..,s) then we get the Lagrange Interpolation polynomials as

$$p_i(\lambda) = \prod_{\substack{1\le j\le k \\ j\ne i}} \frac{(\lambda - \lambda_j)}{(\lambda_i - \lambda_j)}; \quad i = 1,2,\ldots, k \qquad \ldots\ldots\ldots(9)$$

and

$$p_i(\lambda_j) = \delta_{ij} \qquad \ldots\ldots\ldots(10)$$

Now,

$$C_1\phi_1 + C_2\phi_2 + \ldots. + C_k\phi_k = \theta_n$$

For $1 \le i \le k$,

$$p_i(A)[C_1\phi_1 + C_2\phi_2 + \ldots. + C_k\phi_k] = p_i(A)\theta_n = \theta_n$$

$$\Rightarrow C_1 p_i(A)\phi_1 + C_2 p_i(A)\phi_2 + \ldots. + C_k p_i(A)\phi_k = \theta_n$$

$$\Rightarrow C_1 p_i(\lambda_1)\phi_1 + C_2 p_i(\lambda_2)\phi_2 + \ldots. + C_k p_i(\lambda_k)\phi_k = \theta_n, \text{ (by property I on page 86)}$$

$$\Rightarrow C_i\phi_i = \theta; 1 \le i \le k; \text{ by (10)}$$
$$\Rightarrow C_i = 0; 1 \le i \le k \qquad \text{since } \phi_i \text{ are nonzero vectors}$$

Thus

$C_1\phi_1 + C_2\phi_2 + ... + C_k\phi_k = \theta_n \Rightarrow C_1 = C_2 = ... = C_n = 0$ proving (8). Thus we have

Eigen vectors corresponding to distinct eigenvalues of A are linearly independent. PROPERTY II