2.4 SUCCESSIVE OVERRELAXATION (SOR) METHOD

We shall now consider SOR method for the system

$Ax = y$ ………..(I)

We take a scalar parameter $\omega \neq 0$ and multiply both sides of (I) to get an equivalent system,

$\omega Ax = \omega y$ ………………(II)

Now as before we split the given matrx as

$$A = (D + L + U)$$

We write (II) as

$(\omega D + \omega L + \omega U)x = \omega y$,

i.e.

$$(\omega D + \omega L) = - \omega Ux + \omega y$$

i.e.

$$(D + \omega L)x + (\omega-1) Dx = - \omega Ux + \omega y$$

i.e.

$$(D + \omega L)x = - [(\omega - 1)D + \omega U]x + \omega y$$

i.e.

$$x = - (D + \omega L)^{-1} [(\omega-1)D + \omega U]x + \omega [D + \omega L]^{-1} y.$$

We thus get the SOR scheme as

$$\left.\begin{array}{l} x^{(k+1)} = M_{\omega} x^{(k)} + \hat{y} \\ x^{(0)} = \text{zero vector; initial guess} \end{array}\right\} \quad \text{……………(III)}$$

where,

$$M_{\omega} = -(D + \omega L)^{-1} [(\omega - 1)D + \omega u]$$

and

$$\hat{y} = (D + \omega L)^{-1}\omega y$$

M_ω is the SOR matrix for the system.

Notice that if $\omega = 1$ we get the Gauss – Seidel scheme. The strategy is to choose ω such that $\|M_\omega\|_{sp}\, is < 1,$ and is as small as possible so that the scheme converges as rapidly as possible. This is easier said than achieved. How does one choose ω? It can be shown that convergence cannot be achieved if $\omega \geq 2$. (We assume $\omega > 0$). 'Usually' ω is chosen between 1 and 2. Of course, one must analyse $\|M_\omega\|_{sp}$ as a function of ω and find that value ω_0 of ω for which this is minimum and work with this value of ω_0.

Let us consider an example of this aspect.

<u>Example 6:</u>

Consider the system given in example 5 in section 2.3.

For that system,

$M_\omega = -(D + \omega L)^{-1}[(\omega-1)D + \omega U]$

$$= \begin{pmatrix} 1-\omega & \frac{1}{2}\omega & 0 & 0 \\ \frac{1}{2}\omega - \frac{1}{2}\omega^2 & 1-\omega+\frac{1}{4}\omega^2 & \frac{1}{2}\omega & 0 \\ \frac{1}{4}\omega^2 - \frac{1}{4}\omega^3 & \frac{1}{2}\omega - \frac{1}{2}\omega^2 + \frac{1}{8}\omega^3 & 1-\omega+\frac{1}{4}\omega^2 & \frac{1}{2}\omega \\ \frac{1}{8}\omega^3 - \frac{1}{8}\omega^4 & \frac{1}{4}\omega^2 - \frac{1}{4}\omega^3 + \frac{1}{16}\omega^4 & \frac{1}{2}\omega - \frac{1}{2}\omega^2 + \frac{1}{8}\omega^3 & 1-\omega+\frac{1}{4}\omega^2 \end{pmatrix}$$

and the characteristic equation is

$$16(\omega - 1 + \lambda)^4 - 12\omega^2\lambda(\omega - 1 + \lambda)^2 + \omega^4\lambda^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (C_{M\omega})$$

Thus the eigenvalues of M_ω are roots of the above equation. Now when is $\lambda = 0$ a root? If $\lambda = 0$ we get, from $(C_{M\omega})$,

$16(\omega-1)^4 = 0 \Rightarrow \omega = 1$, i.e. as in the Gauss – Seidel case. So let us take $\omega \neq 1$; so $\lambda = 0$ is not a root. So we can divide the above equation $(C_{M\omega})$ by $\omega^4\lambda^2$ to get

$$16\left[\frac{(\omega-1+\lambda)^2}{\omega^2\lambda}\right]^2-12\frac{(\omega-1+\lambda)^2}{\omega^2\lambda}+1=0$$

Setting

$$\mu^2=\frac{(\omega-1+\lambda)^2}{\omega^2\lambda} \quad \text{we get}$$

$$16\mu^4-12\mu^2+1=0$$

which is the same as (C_J). Thus

$$\mu=\pm 0.3090\ ;\pm 0.8090\ .$$

Now

$$\frac{(\omega-1+\lambda)^2}{\omega^2\lambda}=\mu^2=0.0955 \text{ or } 0.6545 \qquad \ldots\ldots\ldots(*)$$

Thus, this can be simplified as

$$\lambda=\frac{1}{2}\mu^2\omega^2-(\omega-1)\pm\mu\omega\left\{\frac{1}{4}\mu^2\omega^2-(\omega-1)\right\}^{\frac{1}{2}}$$

as the eigenvalues of M_ω.

With $\omega = 1.2$ and using the two values of μ^2 in (*) we get,

$\lambda = 0.4545,\ 0.0880,\ -0.1312 \pm i\ (0.1509)$.

as the eigenvalues. The modulus of the complex roots is 0.2

Thus

$\|M_\omega\|_{sp}$ when $\omega = 1.2$ is 0.4545

which is less that $\|J\|_{sp} = 0.8090$ and $\|G\|_{sp} = 0.6545$ computed in Example 5 in section 2.3. Thus for this system, SOR with $\omega = 1.2$ is faster than Jacobi and Gauss – Seidel scheme.

We can show that in this example when ω = ω_0 = 1.2596, the spectral radius $\|M_{\omega 0}\|$ is smaller than $\|M_\omega\|$ for any other ω. We have

$$\|M_{1.2596}\| = 0.2596$$

Thus the SOR scheme with ω = 1.2596 will be the method which converges fastest.

<u>Note:</u>

We had $\|M_{1.2}\|_{sp} = 0.4545$

and

$$\|M_{1.2596}\|_{sp} = 0.2596$$

Thus a small change in the value of ω brings about a significant change in the spectral radius $\|M_\omega\|_{sp}$.