2. ITERATIVE METHODS FOR SOLVING LINEAR SYSTEM OF EQUATIONS

2.1 Introduction To Iterative Methods

In general an iterative scheme is as follows:

We have an nxn matrix M and we want to get the solution of the system

$$x = Mx + y \qquad \text{……………………..(1)}$$

We obtain the solution x as the limit of a sequence of vectors, $\{x^k\}$ which are obtained as follows:

We start with any initial vector $x^{(0)}$, and calculate $x^{(k)}$ from,

$$x^{(k)} = Mx^{(k-1)} + y \qquad \text{……………….(2)}$$

for k = 1,2,3, ….. successively.

We shall mention that a necessary and sufficient condition for the sequence of vectors $x^{(k)}$ to converge to a solution x of (1) is that the spectral radius $\|M\|_{sp}$ of the iterating matrix M is less than 1 or if $\|M\|$ for some matrix norm. (We shall introduce the notion of norm formally in the next unit).

We shall now consider some iterative schemes for solving systems of linear equations,

$$Ax = y \qquad \text{…………….(3)}$$

We write this system in detail as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots.. + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots.. + a_{2n}x_n &= y_2 \\ \ldots\ldots \quad \ldots\ldots \quad & \ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots.. + a_{nn}x_n &= y_n \end{aligned} \qquad \text{. . . . .(4)}$$

We have

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{. . . . . (5)}$$

We denote by D, L, U the matrices

$$D = \begin{pmatrix} a_{11} & 0 & \dots & \dots & 0 \\ 0 & a_{22} & \dots & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a_{nn} \end{pmatrix} \dots\dots\dots\dots\dots\dots(6)$$

the diagonal part of A; and

$$L = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ a_{21} & 0 & \dots & \dots & 0 \\ a_{31} & a_{32} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & 0 \end{pmatrix} \dots\dots\dots\dots\dots\dots(7)$$

the lower triangular part of A; and

$$U = \begin{pmatrix} 0 & a_{12} & \dots & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & a_{n-1,n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \dots\dots\dots\dots\dots\dots(8)$$

the upper triangular part of A.

Note that,

A = D + L + U……………………… (9).

We assume that $a_{ii} \neq 0$; i = 1, 2, ……, n …………(10)

so that D^{-1} exists.

We now describe two important iterative schemes, in the next section, for solving the system (3).