1.2 GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING

In the Gaussian Elimination method discussed in the previous section, at the r^{th} stage we reduced all the entries in the r^{th} column, below the r^{th} principal diagonal entry as zero. In the partial pivoting method, before we do this we look at the entries in the r^{th} diagonal and below it and then pick the one that has the largest absolute value and we bring it to the diagonal position by a row interchange, and then reduce the entries below the r^{th} diagonal as zero. When we incorporate this idea at each stage of the Gaussian elimination process we get the GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING. We now illustrate this with a few examples:

Note: The elementary row operations on the matrix A and the vector y can be simultaneously carried out by introducing the "Augmented matrix", A_{aug} which is obtained by appending y as an additional column at the end.

Example 1:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 + 2x_2 \quad\quad &= 3 \end{aligned}$$

We have

$$A_{aug} = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 1 & 2 \\ 1 & 2 & 0 & 3 \end{array}\right)$$

1st Stage: The pivot has to be chosen as 2 as this is the largest absolute valued entry in the first column. Therefore we do

$$A_{aug} \xrightarrow{R_{12}} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 1 & 2 & 0 & 3 \end{array}\right)$$

Therefore we have

$$M^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } M^{(1)} A^{(1)} = A^{(2)} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$M^{(1)}\,A^{(1)} = y^{(2)} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

Next we have

$$A^{(2)}_{\text{aug}} \xrightarrow[R_3-\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1} \left(\begin{array}{c:cc|c} 2 & -1 & 1 & 2 \\ \hdashline 0 & 3/2 & 3/2 & 3 \\ 0 & 5/2 & -1/2 & 2 \end{array}\right)$$

Here

$$M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \quad ; \qquad M^{(2)}\,A^{(2)} = A^{(3)} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3/2 & 3/2 \\ 0 & 5/2 & -1/2 \end{pmatrix}$$

$$M^2\,y^{(2)} = y^{(3)} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

Now at the next stage the pivot is $\frac{5}{2}$ since this is the entry with the largest absolute value in the 1st column of the next submatrix. So we have to do another row interchange.

Therefore

$$A^{(3)}_{avg} \xrightarrow{R_{23}} \left(\begin{array}{c:cc|c} 2 & -1 & 1 & 2 \\ \hdashline 0 & 5/2 & -1/2 & 2 \\ 0 & 3/2 & 3/2 & 3 \end{array}\right)$$

$$M^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad M^{(3)} A^{(3)} = A^{(4)} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 5/2 & -1/2 \\ 0 & 3/2 & 3/2 \end{pmatrix}$$

$$M^{(3)} y^{(3)} = y^{(4)} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Next we have

$$A_{avg}^{(4)} \xrightarrow{R_2 - \frac{3}{5} R_2} \left(\begin{array}{c|c|c|c} 2 & -1 & 1 & 2 \\ \hline 0 & 5/2 & -1/2 & 2 \\ \hline 0 & 0 & 9/5 & 9/5 \end{array}\right)$$

Here

$$M^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/5 & 1 \end{pmatrix} \qquad M^{(4)} A^{(4)} = A^{(5)} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 5/2 & -1/2 \\ 0 & 0 & 9/5 \end{pmatrix}$$

$$M^{(4)} y^{(4)} = y^{(5)} = \begin{pmatrix} 2 \\ 2 \\ 9/5 \end{pmatrix}$$

This completes the reduction and we have that the given system is equivalent to the system

$A^{(5)}x = y^{(5)}$

i.e.

$$2x_1 - x_2 + x_3 = 2$$

$$\frac{5}{2}x_2 - \frac{1}{2}x_3 = 2$$
$$\frac{9}{5}x_3 = \frac{9}{5}$$

We now get the solution by back substitution:

The 3rd equation gives,

$$x_3 = 1$$

using this in second equation we get

$\frac{5}{2}x_2 - \frac{1}{2} = 2$ giving $\frac{5}{2}x_2 = \frac{5}{2}$ and hence $x_2 = 1$.

Using the values of x_1 and x_2 in the first equation we get

$2x_1 - 1 + 1 = 2$ giving $x_1 = 1$

Thus we get the solution of the system as $x_1 = 1$, $x_2 = 1$, $x_3 = 1$; the same as we had obtained with the simple Gaussian elimination method earlier.

<u>Example 2:</u>
Let us now apply the Gaussian elimination method with partial pivoting to the following example:

$$(0.000003)x_1 + (0.213472)x_2 + (0.332147)\, x_3 = 0.235262$$
$$(0.215512)x_1 + (0.375623)x_2 + (0.476625)\, x_3 = 0.127653$$
$$(0.173257)x_1 + (0.663257)x_2 + (0.625675)\, x_3 = 0.285321,$$

the system to which we had earlier applied the simple GEM and had obtained solutions which were far away from the correct solutions.

Note that

$$A = \begin{pmatrix} 0.000003 & 0.213472 & 0.332147 \\ 0.215512 & 0.375623 & 0.476625 \\ 0.173257 & 0.663257 & 0.625675 \end{pmatrix}$$

$$y = \begin{pmatrix} 0.235262 \\ 0.127653 \\ 0.285321 \end{pmatrix}$$

We observe that at the first stage we must choose 0.215512 as the pivot. So we have

$$A^{(1)} = A \xrightarrow{R_{12}} A^{(2)} = \begin{pmatrix} 0.215512 & 0.375623 & 0.476625 \\ 0.000003 & 0.213472 & 0.332147 \\ 0.173257 & 0.663257 & 0.625675 \end{pmatrix}$$

$$y^{(1)} = y \xrightarrow{R_{12}} y^{(2)} = \begin{pmatrix} 0.127653 \\ 0.235262 \\ 0.285321 \end{pmatrix} \qquad M^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next stage we make all entries below 1st diagonal as zero

$$A^{(2)} \xrightarrow[R_3+m_{21}R_1]{R_2+m_{21}R_1} A^{(3)} = \left(\begin{array}{c|cc} 0.215512 & 0.375623 & 0.476625 \\ \hline 0 & 0.213467 & 0.332140 \\ 0 & 0.361282 & 0.242501 \end{array}\right)$$

where

$$m_{21} = -\frac{a_{21}}{a_{11}} = -\frac{0.000003}{0.215512} = -0.000014$$

$$m_{31} = -\frac{a_{31}}{a_{11}} = -\frac{0.173257}{0.215512} = -0.803932$$

$$M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -0.000014 & 1 & 0 \\ -0.803932 & 0 & 1 \end{pmatrix}; \qquad y^{(3)} = M^{(2)} y^{(2)} = \begin{pmatrix} 0.127653 \\ 0.235260 \\ 0.182697 \end{pmatrix}$$

In the next stage we observe that we must choose 0.361282 as the pivot. Thus we have to interchange 2nd and 3rd row. We get,

$$A^{(3)} \xrightarrow{R_{23}} A^{(4)} = \begin{pmatrix} 0.215512 & 0.375623 & 0.476625 \\ 0 & 0.361282 & 0.242501 \\ 0 & 0.213467 & 0.332140 \end{pmatrix}$$

$$M^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad y^{(4)} = M^{(3)} y^{(3)} = \begin{pmatrix} 0.127653 \\ 0.182697 \\ 0.235260 \end{pmatrix}$$

Now reduce the entry below 2nd diagonal as zero

$$A^{(4)} \xrightarrow{R_3 + m_{32} R_2} A^5 = \begin{pmatrix} 0.215512 & 0.375623 & 0.476625 \\ 0 & 0.361282 & 0.242501 \\ 0 & 0 & 0.188856 \end{pmatrix}$$

$$m_{32} = -\frac{0.213467}{0.361282} = -0.590860$$

$$M^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.59086 & 1 \end{pmatrix} \qquad y^{(5)} = M^{(4)} y^{(4)} = \begin{pmatrix} 0.127653 \\ 0.182697 \\ 0.127312 \end{pmatrix}$$

Thus the given system is equivalent to

$A^{(5)} x = y^{(5)}$

which is an upper triangular system and can be solved by back substitution to get

$$\left.\begin{aligned} x_3 &= 0.674122 \\ x_2 &= 0.053205 \\ x_1 &= -0.991291 \end{aligned}\right\} ,$$

which compares well with the 10 decimal accurate solution given at the end of section 1.1(page11). Notice that while we got very bad errors in the solutions while using simple GEM whereas we have come around this difficulty by using partial pivoting.