LINEAR SYSTEMS OF EQUATIONS
AND
MATRIX COMPUTATIONS

1. DIRECT METHODS FOR SOLVING LINEAR SYSTEMS OF EQUATIONS

1.1 SIMPLE GAUSSIAN ELIMINATION METHOD

Consider a system of n equations in n unknowns,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2\\
\ldots \quad \ldots \quad \ldots \quad &\ldots \quad \ldots\\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n
\end{aligned}
$$

We shall assume that this system has a unique solution and proceed to describe the simple "Gaussian Elimination Method", (from now on abbreviated as GEM),Page 2 of 11 for finding the solution. The method reduces the system to an upper triangular system using elementary row operations (ERO).

Let $A^{(1)}$ denote the coefficient matrix A.

$$
A^{(1)} = \begin{pmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & \ldots\ldots & a_{1n}^{(1)}\\
a_{21}^{(1)} & a_{22}^{(1)} & \ldots\ldots & a_{2n}^{(1)}\\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\
a_{n1}^{(1)} & a_{n2}^{(1)} & \ldots\ldots & a_{nn}^{(1)}
\end{pmatrix}
$$

where $a_{ij}^{(1)} = a_{ij}$

Let

$$
y^{(1)} = \begin{pmatrix} y_1^{(1)}\\ y_2^{(1)}\\ \vdots\\ y_n^{(1)} \end{pmatrix}
$$

where $y_i^{(1)} = y_i$

We assume $a_{11}^{(1)} \neq 0$

Then by ERO applied to $A^{(1)}$, (that is, subtracting suitable multiples of the first row from the remaining rows), reduce all entries below $a_{11}^{(1)}$ to zero. Let the resulting matrix be denoted by $A^{(2)}$.

$$
A^{(1)} \xrightarrow{R_i + m_{i1}^{(1)} R_1} A^{(2)}
$$

where $\quad m_{i1}^{(1)} = -\dfrac{a_{i1}^{(1)}}{a_{11}^{(1)}}; \quad i > 1.$

Note $A^{(2)}$ is of the form

$$A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & \dots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & \dots & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2}^{(2)} & \dots & \dots & a_{nn}^{(2)} \end{pmatrix}$$

Notice that the above row operations on $A^{(1)}$ can be effected by premultiplying $A^{(1)}$ by $M^{(1)}$ where

$$M^{(1)} = \left(\begin{array}{c|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline m_{21}^{(1)} & & & & & \\ m_{31}^{(1)} & & I_{n-1} & & & \\ \vdots & & & & & \\ m_{n1}^{(1)} & & & & & \end{array}\right)$$

(I_{n-1} being the n-1 $\times$ n-1 identity matrix).
i.e.

$$M^{(1)} A^{(1)} = A^{(2)}$$

Let

$$y^{(2)} = M^{(1)} y^{(1)}$$

i.e.

$$y^{(1)} \xrightarrow{R_i + m_{i1} R_1} y^{(2)}$$

Then the system Ax = y is equivalent to

$A^{(2)}x = y^{(2)}$

Next we assume

$a_{22}^{(2)} \neq 0$

and reduce all entries below this to zero by ERO

$$A^{(2)} \xrightarrow{R_i + m_{i2}^{(2)}} A^{(3)} \;;$$

$$m_{i2}^{(2)} = -\frac{a_{i2}^{(2)}}{a_{22}^{(2)}}; \qquad i > 3$$

Here

$$M^{(2)} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & m_{32}^{(2)} & & & \\ 0 & m_{42}^{(2)} & & I_{n-2} & \\ \vdots & \vdots & & & \\ 0 & m_{n2}^{(2)} & & & \end{array}\right)$$

and

$M^{(2)} A^{(2)} = A^{(3)}$; $\qquad$ $M^{(2)} y^{(2)} = y^{(3)}$;

and $A^{(3)}$ is of the form

$$A^{(3)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & \ldots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \ldots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \ldots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \ldots & a_{nn}^{(3)} \end{pmatrix}$$

We next assume $a_{33}^{(3)} \neq 0$ and proceed to make entries below this as zero. We thus get $M^{(1)}$, $M^{(2)}$, …. , $M^{(r)}$ where

$$M^{(r)} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & 0_{r\times(n-r)} \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & \\ \hline 0 & 0 & \cdots & m_{r+1r}^{(r)} & \\ 0 & 0 & \cdots & m_{r+2r}^{(r)} & I_{n-r} \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & m_{nr}^{(r)} & \end{array}\right)$$

$$M^{(r)}A^{(r)} = A^{(r+1)} = \begin{pmatrix} a_{11}^{(1)} & \ldots & \ldots & \ldots & \ldots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ldots & \ldots & \ldots & a_{2n}^{(r)} \\ \vdots & 0 & a_{rr}^{(r)} & \ldots & \ldots & a_{rn}^{(r)} \\ \vdots & \vdots & 0 & a_{r+1r+1}^{(r+1)} & \ldots & a_{r+1n}^{(r+1)} \\ \vdots & \vdots & \vdots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & a_{nr+1}^{(r+1)} & \ldots & a_{nn}^{(r+1)} \end{pmatrix}$$

$M^{(r)}\, y^{(r)} = y^{(r+1)}$

At each stage we assume $a_{rr}^{(r)} \neq 0$.

Proceeding thus we get,

$M^{(1)}, M^{(2)}, \ldots, M^{(n-1)}$ such that

$M^{(n-1)}\, M^{(n-2)} \ldots M^{(1)}\, A^{(1)} = A^{(n)}$; $M^{(n-1)}\, M^{(n-2)} \ldots M^{(1)}\, y^{(1)} = y^{(n)}$

where
$$A^{(n)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & & \ddots & \\ & & & a_{nn}^{(n)} \end{pmatrix}$$

which is an upper triangular matrix and the given system is equivalent to

$A^{(n)}x = y^{(n)}$

Since this is an upper triangular, this can be solved by back substitution; and hence the system can be solved easily.

Note further that each $M^{(r)}$ is a lower triangular matrix with all diagonal entries as 1. Thus determinant of $M^{(r)}$ is 1 for every r. Now,

$A^{(n)} = M^{(n-1)} \ldots M^{(1)}\, A^{(1)}$

Thus

$\det A^{(n)} = \det M^{(n-1)}\ \det M^{(n-2)} \ldots \det M^{(1)}\ \det A^{(1)}$

$\det A^{(n)} = \det A^{(1)} = \det A$ since $A = A^{(1)}$

Now $A^{(n)}$ is an upper triangular matrix and hence its determinant is $a_{11}^{(1)} a_{22}^{(2)} \cdots a_{nn}^{(n)}$. Thus det A is given by

$$\det A = a_{11}^{(1)} a_{22}^{(2)} \cdots a_{nn}^{(n)}$$

Thus the simple GEM can be used to solve the system Ax = y and also to evaluate det A provided $a_{ii}^{(i)} \neq 0$ for each i.
Further note that $M^{(1)}$, $M^{(2)}$, …. , $M^{(n-1)}$ are lower triangular, and nonsingular as their det = 1 and hence not zero. They are all therefore invertible and their inverses are all lower triangular, i.e. if $\mathcal{L} = M^{(n-1)} M^{(n-2)} \ldots M^{(1)}$ then $\mathcal{L}$ is lower triangular and nonsingular and $\mathcal{L}^{-1}$ is also lower triangular.

Now $\mathcal{L}A = \mathcal{L}A^{(1)} = M^{(n-1)} M^{(n-2)} \ldots M^{(1)} A^{(1)} = A^{(n)}$

Therefore $A = \mathcal{L}^{-1} A^{(n)}$

Now $\mathcal{L}^{-1}$ is lower triangular which we denote by L and $A^{(n)}$ is upper triangular which we denote by U, and we thus get the so called LU decomposition

A = LU

of a given matrix A – as a product of a lower triangular matrix with an upper triangular matrix. This is another application of the simple GEM. REMEMBER IF AT ANY STAGE WE GET $a_{ii}^{(1)} = 0$ WE CANNOT PROCEED FURTHER WITH THE SIMPLE GEM.

EXAMPLE:

Consider the system

$x_1 + x_2 + 2x_3 = 4$
$2x_1 - x_2 + x_3 = 2$
$x_1 + 2x_2 \quad\quad = 3$

Here

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad\qquad y = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \underset{R_3 - R_1}{\xrightarrow{R_2 - 2R_1}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 1 & -2 \end{pmatrix} = A^{(2)}$$

$$a_{11}^{(1)} = 1 \neq 0$$

$$\left.\begin{array}{l} m_{21}^{(1)} = -2 \\ m_{31}^{(1)} = -1 \end{array}\right\}$$

$$a_{22}^{(2)} = -3 \neq 0$$

$$M^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad y^{(1)} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \to \begin{pmatrix} 4 \\ -6 \\ -1 \end{pmatrix} = y^{(2)} = M^{(1)} y^{(1)}$$

$$A^{(2)} \xrightarrow{R_3 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{pmatrix} = A^{(3)} \qquad a_{33}^{(3)} = -3$$

$$m_{31}^{(2)} = 1/3$$

$$M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix} \qquad y^{(3)} = M^{(2)} y^{(2)} = \begin{pmatrix} 4 \\ -6 \\ -3 \end{pmatrix}$$

Therefore the given system is equivalent to $A^{(3)}x = y^{(3)}$,i.e.,

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \\ -3x_2 - 3x_3 &= -6 \\ -3x_3 &= -3 \end{aligned}$$

<u>Back Substitution</u>

$x_3 = 1$

$-3x_2 - 3 = -6 \Rightarrow -3x_2 = -3 \Rightarrow x_2 = 1$

$x_1 + 1 + 2 = 4 \Rightarrow x_1 = 1$

Thus the solution of the given system is,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The determinant of the given matrix A is

$$a_{11}^{(1)} a_{22}^{(2)} a_{33}^{(3)} = (1)(-3)(-3) = 9.$$

Now

$$\left(M^{(1)}\right)^{(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\left(M^{(2)}\right)^{(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix}$$

$\mathscr{L} = M^{(2)} M^{(-1)}$

$$\mathscr{L}^{-1} = \left(M^{(2)} M^{(1)}\right)^{-1} = \left(M^{(1)}\right)^{-1} \left(M^{(2)}\right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix}$$

$$L = \mathscr{L}^{(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/3 & 1 \end{pmatrix}$$

$$U = A^{(n)} = A^{(3)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{pmatrix}$$

Therefore A = LU   i.e.,

$$A=\begin{pmatrix}1 & 1 & 2\\ 2 & -1 & 1\\ 1 & 2 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 1 & -1/3 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 0 & -3 & -3\\ 0 & 0 & -3\end{pmatrix}$$

is the LU decomposition of the given matrix A.

We observed that in order to apply simple GEM we need $a_{rr}^{(r)} \neq 0$ for each stage r. This may not be satisfied always. So we have to modify the simple GEM in order to overcome this situation. Further, even if the condition $a_{rr}^{(r)} \neq 0$ is satisfied at each stage, simple GEM may not be a very accurate method to use. What do we mean by this? Consider, as an example, the following system:

$$(0.000003)\, x_1 + (0.213472)\, x_2 + (0.332147)\, x_3 = 0.235262$$
$$(0.215512)\, x_1 + (0.375623)\, x_2 + (0.476625)\, x_3 = 0.127653$$
$$(0.173257)\, x_1 + (0.663257)\, x_2 + (0.625675)\, x_3 = 0.285321$$

Let us do the computations to 6 significant digits.

Here,

$$A^{(1)} = \begin{pmatrix}0.000003 & 0.213472 & 0.332147\\ 0.215512 & 0.375623 & 0.476625\\ 0.173257 & 0.663257 & 0.625675\end{pmatrix}$$

$$y^{(1)} = \begin{pmatrix}0.235262\\ 0.127653\\ 0.285321\end{pmatrix} \qquad a_{11}^{(1)} = 0.000003 \neq 0$$

$$m_{21}^{(1)} = -\frac{a_{21}^{(1)}}{a_{11}^{(1)}} = -\frac{0.215512}{0.000003} = -71837.3$$

$$m_{31}^{(1)} = -\frac{a_{31}^{(1)}}{a_{11}^{(1)}} = -\frac{0.173257}{0.000003} = -57752.3$$

$$M^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -71837.3 & 1 & 0 \\ -57752.3 & 0 & 1 \end{pmatrix} ; \qquad y^{(2)} = M^{(1)} y^{(1)} = \begin{pmatrix} 0.235262 \\ -16900.5 \\ -13586.6 \end{pmatrix}$$

$$A^{(2)} = M^{(1)} A^{(1)} = \begin{pmatrix} 0.000003 & 0.213472 & 0.332147 \\ 0 & -15334.9 & -23860.0 \\ 0 & -12327.8 & -19181.7 \end{pmatrix}$$

$$a_{22}^{(2)} = -15334.9 \neq 0$$

$$m_{32}^{(2)} = -\frac{a_{32}^{(2)}}{a_{22}^{(2)}} = -\frac{-12327.8}{-15334.9} = -0.803905$$

$$M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.803905 & 1 \end{pmatrix}$$

$$y^{(3)} = M^{(2)} y^{(2)} = \begin{pmatrix} 0.235262 \\ -16900.5 \\ -0.20000 \end{pmatrix}$$

$$A^{(3)} = M^{(2)} A^{(2)} = \begin{pmatrix} 0.000003 & 0.213472 & 0.332147 \\ 0 & -15334.9 & -23860.0 \\ 0 & 0 & -0.50000 \end{pmatrix}$$

Thus the given system is equivalent to the upper triangular system

$A^{(3)}x = y^{(3)}$

Back substitution yields,

$$\left.\begin{aligned} x_3 &= 0.40\,00\,00 \\ x_2 &= 0.47\,97\,23 \\ x_1 &= -1.33\,33\,3 \end{aligned}\right\}$$

This compares poorly with the correct answers (to 10 digits) given by

$$\left.\begin{aligned} x_1 &= 0.67\ 41\ 21\ 46\ 94 \\ x_2 &= 0.05\ 32\ 03\ 93\ 39.1 \\ x_3 &= -0.99\ 12\ 89\ 42\ 52 \end{aligned}\right\}$$

Thus we see that the simple Gaussian Elimination method needs modification in order to handle the situations that may lead to $a_{rr}^{(r)} = 0$ for some r or situations as arising in the above example. In order to do this we introduce the idea of **Partial Pivoting** in the next section.