Introduction to Nonlinear Dynamics
Indian Institute of Technology Madras
NPTEL September 2015

Selected Exercises 1

September 24, 2015

Disclaimer: Please note that this set of "Selected Exercises" is an extension of the assignment sheet, and contains the assignment questions as a subset. However, unlike the assignment sheet, this should *not* be turned in! Also, this is not an exhaustive list of exercises. In fact, it is a subset of the exercises given in the book "Nonlinear Dynamics and Chaos", First Indian Edition (2007) by Steven H. Strogatz. Interested students are encouraged to solve all the exercise problems from the aforementioned book.

1 Geometric Intuition

1) [*Mechanical analog*]

   a) Find a mechanical system that is approximately governed by $\dot{x} = \sin x$.

   b) Using your physical intuition, explain why it now becomes obvious that $x^* = 0$ is an unstable fixed point and $x^* = \pi$ is stable.

2) [*Exact solution*] The system $\dot{x} = \sin x$ has the solution $t = \ln|\,(\csc x_0 + \cot x_0)/(\csc x + \cot x)\,|$, where $x_0 = x(0)$ is the initial value of x.

   a) Given the specific initial condition $x_0 = \pi/4$, show that the solution described above can be inverted to obtain

   $$x(t) = 2\,\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).$$

   Conclude that $x(t) \to \pi$ as $t \to \infty$.

   b) Find the analytical solution for any arbitrary initial condition x_0.

2 Fixed points and stability

1) [*Analyse graphically*] For the following systems, sketch the vector field on the real line, find all the fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions.

   a) $\dot{x} = e^{-x} \sin x$

   b) $\dot{x} = e^{x} - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You will not be able to find the fixed points explicitly, but you can still find the qualitative behaviour.)

2) [*Fixed points*] For each of the following (a)-(c), find an equation $\dot{x} = f(x)$ ($f(x)$ being a smooth function) with the stated properties, or if there are no examples, explain why not.

   a) Every real number is a fixed point.

   b) Every integer is a fixed point, and there are no others.

   c) There are precisely 100 fixed points.

3) [*Working backwards*] Draw the phase portrait for a system that has 3 fixed points: $x = -1$ (stable from left and unstable from right), $x = 0$ (stable from both sides) and $x = 2$ (unstable from both sides). From the phase portrait, find an equation of the form $\dot{x} = f(x)$ that is consistent with it.

3 Linear stability analysis

1) Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

   a) $\dot{x} = \tan x$

   b) $\dot{x} = \ln x$

   c) $\dot{x} = ax - x^3$, discuss three cases where a is positive, negative, or zero.

2) [*Critical slowing down*] In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

   a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but the decay is not exponential. (You should find that the decay is a much slower algebraic function of t.)

   b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \le t \le 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.