1. Design a oil mug, shown in fig.1, to hold as much oil possible. The height and radius of the mug should not be more than 16cm. The mug must be at least 6cm in radius. The surface area of the sides must not be greater than \(800\text{ cm}^2\) (ignore the area of the bottom of the mug and ignore the mug handle). Formulate the optimum design problem.

![Figure 1: Oil mug](image)

2. A refinery has two crude oils:

   (a) Crude A cost Rs. 1800 per barrel and 15000 barrel are available.

   (b) Crude B costs Rs. 2200 per barrel and 25000 barrel are available.

The company manufactures gasoline and lube oil from the crude oils. Yield and sale price per barrel of the product and markets are shown in Table. How much crude oils should the company use to maximize its profit? Formulate the optimum design problem.

<table>
<thead>
<tr>
<th>Product</th>
<th>Yield/barrel Crude A</th>
<th>Yield/barrel Crude B</th>
<th>Sale Price per bbl(Rs.)</th>
<th>Market (bbl)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gasoline</td>
<td>0.5</td>
<td>0.7</td>
<td>2800</td>
<td>20000</td>
</tr>
<tr>
<td>Lube oil</td>
<td>0.5</td>
<td>0.3</td>
<td>7500</td>
<td>8000</td>
</tr>
</tbody>
</table>

3. A company manufactures two types of products X and Y, sells them at a profit of Rs.8 type on X and Rs.10 on type Y. Each product is processed on two machines A and B. Type X (product) requires, 4 minutes of processing time on A (machine) and 6 minutes on B type. Type Y (product) requires 4 minutes on A (machine) and 4 minutes on B. The machine, A is available for not more than 6 hours 30 minutes, while machine B is available for 10 hours during any working day. Maximize the profit on the products X and Y where \(x_1 =\) number of products of type X and \(x_2 =\) number of products of type Y. Formulate the problem as a linear optimization problem.

4. Two electric generators are interconnected to provide total power to meet the load. Each generator’s cost is a function of the power output, as shown in fig.2. All costs and power are expressed on a per unit basis. The total power needed is at least 60 units. Formulate a minimum cost design problem to determine the power outputs \(P_1\) and \(P_2\).
5. Consider the standard bridge circuit shown in fig.3. The power delivered by the batteries is $P_b$ and power consumed by the resistors is $P_r$. Formulate the solution to the optimization problem of minimizing $P_r$ with respect to the currents $(i_1, i_2, i_3)$ and subject to the constraints that will yield the Kirchhoff’s loop equations.

6. The shortest length ‘$l$’ between a given point $x^0 = [x_1^0 \ x_2^0]^T$ and a given line $a_0 + a_1 x_1 + a_2 x_2 = 0$ is expressed as $l = \frac{|a^T x^0 - b|}{|a^T a|}$. Derive the expression by formulating the problem in optimization framework where $a = [a_1 \ a_2]^T$, $b = -a_0$ and $x = [x_1 \ x_2]^T$ is any point on the line.

7. Determine the dimensions for building a minimum cost cylindrical refrigeration tank of volume $60 \ m^3$, if the cylindrical ends cost Rs.700.0/m$^2$, cylindrical wall costs Rs.450.0/mm$^2$, and it costs Rs.600.0/mm$^2$ to refrigerate over the useful life.

8. Sketch the feasible space on a graph paper for each of the following independent constraints, given that $x_1, x_2 \geq 0$

   $$(i) -3x_1 + x_2 \leq 6 \quad (ii)x_1 - 2x_2 \geq 5$$

9. Sketch the feasible solution space of the following optimization problem

   \[
   \text{Maximize } f(x_1, x_2) = f(x) = 5x_1 + 4x_2 \\
   \text{subject to } 6x_1 + 4x_2 \leq 24
   \]
\[ x_1 + 2x_2 \leq 6 \]
\[ -x_1 + x_2 \leq 1 \]
\[ x_2 \leq 2 \]
\[ x_1, x_2 \geq 0 \]

Using graphical method and hence obtain the maximum value of the function \( f(x_1, x_2) \).

10. Find the stationary points for the following functions.

(a) \( f(x) = 10x^2 - 28.6x + 22 \)
(b) \( f(x_1, x_2) = 3x_1^2 + 2x_2^2 + 2x_1x_2 + 7 \)
(c) \( f(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2 \)
(d) \( f(x_1, x_2) = 3x_1^2 + 5x_2^2 - 2x_1x_2 + 8x_2 \)
(e) \( f(x_1, x_2) = -3x_1^2 - 5x_2^2 + 2x_1x_2 - 8x_2 \)
(f) \( f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 - 2x_2x_3 \)
(g) \( f(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 3x_3^2 - x_1x_2 + 2x_2x_3 \)

Also determine the local minimum, local maximum, and inflection (saddle) points (inflection or saddle points are those points that are neither minimum nor maximum). Find the function values around the stationary points and plot them. Comments whether the following functions are unimodal around the stationary points.

11. Suppose \( f(x) \) is a convex function, and \( a, b \in dom f \) with \( a < b \)

(a) Show that
\[ f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \]
for all \( x \in [a, b] \).

(b) Show that
\[ \frac{f(x) - f(a)}{x-a} \leq \frac{f(b) - f(a)}{b-a} \leq \frac{f(b) - f(x)}{b-x} \]
for all \( x \in [a, b] \). Draw a sketch that illustrates this inequality.

(c) Suppose \( f(x) \) is differentiable. Use the result in 12 (b) to show that
\[ \dot{f}(x) \leq \frac{f(b) - f(a)}{b-a} \leq \dot{f}(b) \]

(d) Suppose \( f(x) \) is twice differentiable. Use the result in 12 (c) to show that
\[ \dot{f}(a) \geq 0 \text{ and } \dot{f}(b) \geq 0 \]

12. Determine the nature (i.e -ve definite (semi-definite) function, or +ve definite (semi-definite) function, or indefinite function) of the following quadratic forms:

(a) \( f(x) = x_1^2 + 4x_1^2x_2 + 2x_1x_2 - 7x_2^2 - 6x_2x_3 + 5x_3^2 \)
(b) \( f(x) = x_1^2 - x_2^2 - 2x_2x_3 + x_3^2 \)
(c) \( f(x) = 2x_1^2 + 2x_2^2 - 2x_1x_3 + 3x_3^2 + x_1x_2 \)
13. Consider the following problem with equality constraints:

\[
\text{Minimize } (x_1 - 1)^2 + (x_2 - 1)^2 \\
\text{subject to } x_1 + x_2 = 4 \\
x_1 - x_2 = 0
\]

(a) Is it a valid optimization problem? Explain.
(b) Explain how you would solve the problem? Are necessary conditions needed to find the optimum solution?

14. Solve the following problems graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

(a) Minimize \( f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 \) subject to \( x_1 + x_2 = 4 \)
(b) Minimize \( f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 \) subject to \( x_1 + x_2 \leq 4 \)
(c) Minimize \( f(x_1, x_2) = 3x_1^2 + 5x_2^2 - 2x_1x_2 + 8x_2 \) subject to \( x_1^2 - x_2^2 + 8x_2 \leq 16 \)

15. Show that a general form for a quadratic function is \( f(x_{n \times 1}) = x^T P x + b^T x + c \) where, \( C \) is a scalar and \( b \) is an \( n \times 1 \) vector.

16. Compute the optimum values of the function \( f(x) = 5x^6 - 36x^5 + \frac{165}{2} x^4 - 60x^3 + 36 \)

17. Consider the

\[
f(x_{2 \times 1}) = 2x_1^2 + 4x_1x_2 + x_2^2 + 3x_1 + 4x_2 + 7 = x^T P x + b^T x + c
\]

where, \( x = [x_1, x_2]^T \)

(a) Find the matrix \( P_{2 \times 2} \), vector \( b_{2 \times 1} \) and the scalar quantity \( c \).
(b) Check the nature of the function.
(c) Find all points that satisfy the first-order necessary condition for \( f(x) \) to be optimal.
(d) Does the function \( f(x) \) have a minimizer? If it does, then find all minimizer(s); otherwise explain why it does not.

18. Consider the following functions:

(a) minimize \( f(x_1, x_2) = 3x_1^2 - 4x_1x_2 + 2x_2^2 + 4x_1 + 6 \) with starting point \( x_1^{(0)} = 0.0, \text{ and } x_2^{(0)} = 0.0 \)
(b) minimize \( f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 \) with starting point \( x_1^{(0)} = 1.0, \text{ and } x_2^{(0)} = 0.5 \)
(c) maximize \( f(x_1, x_2) = -x_1^2 + x_1x_2 - 3x_2^2 \) with starting point \( x_1^{(0)} = 1.0, \text{ and } x_2^{(0)} = 2 \)
(d) minimize \( f(x_1, x_2) = 12.1x_1^2 - 1.73x_1 - x_2 + 21.5x_2^2 \) with starting point \( x_1^{(0)} = 1.0, \text{ and } x_2^{(0)} = 1 \)
(e) minimize \( f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 2x_2x_3 + 2x_2^2 + 2x_3^2 \) with starting point \( x_1^{(0)} = 1.0, \text{ and } x_2^{(0)} = 1 \) and \( x_3 = 1.0 \)

using (i) Steepest Descent Method (ii) Conjugate Gradient Method (iii) Newton’s Method. For each method show only three iterations and hence make your comments.

19. Find the points satisfying KKT necessary conditions for the followings, Check if they are optimum points using the graphical method (if possible).
(a) Minimize \( f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1; \)
subject to \( x_1 + x_2 = 4 \)
(b) Minimize \( f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2; \)
subject to \( x_1 + x_2 = 4 \)
(c) Maximum \( f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8; \)
subject to \( x_1 + x_2 = 4 \)
(d) Minimize \( f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2; \)
subject to \( x_1 + x_2 \leq 4 \) and \( x_1 - 3x_2 = 1 \)
(e) Minimize \( f(x_1, x_2) = 3x_1^2 + 5x_2^2 - 2x_1x_2 + 8x_1; \)
subject to \( x_1^2 - x_2^2 + 8x_2 \leq 16 \)

20. Consider the circuit in fig.4 Formulate and solve the KKT for the following problems:( Assume \( i \) is the current following through the circuit.)

![Resistive Circuit Diagram](image)

**Figure 4: A resistive circuit**

(a) Find the value of the resistance \( R \geq 0 \) such that the power absorbed by this resistor is maximized.
(b) Find the value of the resistance \( R \geq 0 \) such that the power delivered to the 10Ω resistor is maximized.
21. Convert the following problems to the standard linear programming (LP) problem.

(a) Maximize
\[ f = x_1 + 2x_2 \]
subject to
\[ -x_1 + 3x_2 \leq 10 \]
\[ x_1 - x_2 \leq 2 \]
\[ x_1 + x_2 \leq 6 \]
\[ x_1 + 3x_2 \geq 6 \]
\[ x_1, x_2 \geq 0 \]

(b) Minimize
\[ f = x_1 + 4x_2 \]
subject to
\[ x_1 + 2x_2 \leq 5 \]
\[ 2x_1 + x_2 = 4 \]
\[ x_1 - x_2 \geq 3 \]
\[ x_1 \geq 0, x_2 \text{ unrestricted in sign} \]

22. Solve the following problems by the Simplex method and verify the solution graphically whenever possible.

(a) Maximize
\[ f = x_1 + 0.5x_2 \]
subject to
\[ 6x_1 + 5x_2 \leq 30 \]
\[ 3x_1 + x_2 \leq 12 \]
\[ x_1 + 3x_2 \leq 12 \]
\[ x_1, x_2 \geq 0 \]

(b) Maximize
\[ f = 5x_1 - 2x_2 \]
subject to
\[ 2x_1 + x_2 \leq 9 \]
\[ x_1 - x_2 \leq 2 \]
\[ -3x_1 + 2x_2 \leq 3 \]
\[ x_1, x_2 \geq 0 \]

(c) Minimize
\[ f = 5x_1 + 4x_2 - x_3 \]
subject to
\[ x_1 + 2x_2 - x_3 \geq 1 \]
\[ 2x_1 + x_2 + x_3 \geq 4 \]
\[ x_1, x_2 \geq 0; x_3 \text{ is unrestricted in sign} \]

(d) Maximize
\[ f = x_1 + 4x_2 \]
subject to
\[ x_1 + 2x_2 \leq 5 \]
\[ 2x_1 + x_2 = 4 \]
\[ x_1 - x_2 \geq 1 \]
\[ x_1, x_2 \geq 0; \]
23. Write the dual problems for the following problems and solve the dual the problems and recover the values of the primal variables from the final dual tableau.

(a) Maximize \( z(x_1, x_2) = x_1 + 4x_2 \)
subject to \( x_1 + 2x_2 \leq 5 \)
\( x_1 + x_2 = 4 \)
\( x_1 - x_2 \geq 3 \)
\( x_1, x_2 \geq 0 \)

(b) Minimize \( f(x_1, x_2) = 20x_1 - 6x_2 \)
subject to \( 3x_1 - x_2 \geq 3 \)
\( -4x_1 + 3x_2 = -8 \)
\( x_1, x_2 \geq 0 \)

24. Consider the problem of

\[
\text{Minimize } f(x_1, x_2) = x_1^2 + 4x_2^2 - 8x_1 - 16x_2 ,
\]
subject to \( x_1 + x_2 \leq 5 \),
\( x_2 \leq 3 \),
\( x_1 \geq 0; \quad x_2 \geq 0 \)

(a) Is this problem a quadratic programming problem (QPP)? Justify your answer.
(b) Express the objective function in standard QPP form.
(c) Convert the above QPP into an equivalent standard linear programming problem.

25. Solve the following Quadratic programming problems using KKT condition (see lecture notes-22)

(a) Minimize \( f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2 - 2x_2 + 2 \)
subject to \( x_1 + x_2 \leq 4 \),
\( x_1 \geq 0; \quad x_2 \geq 0 \)

(b) Minimize \( f(x_1, x_2) = 2x_1^2 + 9x_2^2 + 6x_1x_2 - 18x_1 + 9x_2 \)
subject to \( x_1 - 2x_2 \leq 10 \),
\( 4x_1 - 3x_2 \leq 20 \)
\( x_1 \geq 0; \quad x_2 \geq 0 \)
Assignment-3 (Related to Lecture notes 23-29)

26. Solve the problems,

(a) Maximize $f(x_1, x_2) = x_1 + 2x_2$, 
subject to $x_1 + x_2 \leq 8$, 
$x_1 \geq 0; \quad x_2 \geq 0$

(b) Maximize $f(x_1, x_2) = -4x_1 + 2x_2$, 
subject to $x_1 + x_2 \leq 10$, 
$x_2 \leq 5$ 
$x_1 \geq 0; \quad x_2 \geq 0$

using interior point method based on centering scheme. Assume the initial trial solution for (a) as $x^0 = [2 \ 2 \ 4]^T$ and for (b) as $x^0 = [3 \ 3 \ 4 \ 2]^T$. Show only one iteration.

27. Solve the following problems

(a) Minimize $f(x) = x_1^2 + x_2$, 
subject to $x_1^2 + x_2^2 - 1 \leq 0$, 

using the logarithm barrier function method (interior penalty method). Apply (i) Analytical implementation method (ii) Numerical implementation method with $\lambda_{k+1} = 0.1\lambda_k$, for $k = 0, 1, 2$. Assume $\lambda_0 = 0.1$.

(b) Minimize $f(x) = 3x_1^2 + x_2^2$; where $x = [x_1 \ x_2]^T$ 
subject to $g(x) = 2 - x_1 - x_2 \leq 0$; $x_1 > 0$ and $x_2 > 0$ 

applying the logarithmic interior penalty method and hence estimate Lagrange multiplier using the expression $\mu^* = \lim_{k \to \infty} \frac{-\tau_k}{g}$; where $\tau_k$ is the penalty coefficient.

28. Solve the following problems

(a) Minimize $f(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 4)^2$; 
subject to $g(x) = 5 - x_1 - x_2 \geq 0$

using exterior point method. Show the calculations only for two iterations, the penalty coefficients values at two successive iterations are considered as $\tau_0 = 1$ and $\tau_1 = 10$ respectively.

(b) Minimize $f(x) = 3x_1^2 + x_2^2$; where $x = [x_1 \ x_2]^T$ 
subject to $g(x) = 2 - x_1 - x_2 \leq 0$; $x_1 > 0$ and $x_2 > 0$. 

using the exterior point method.
Assignment-4 (Related to Lecture notes 30-52)

29. Find the variations of the functionals:
   (a) \( v(x_1(t), x_2(t)) = \int_{0}^{\pi/2} (x_1^4(t) + 2x_1^2(t)x_2^2(t) + x_2^2(t)) dt \),
   (b) \( v(x_1(t), x_2(t)) = \int_{0}^{1} (e^{(x_1(t)-x_2(t))} - e^{(x_1(t)+x_2(t))}) dt \).

30. Find the equations of the curves that are extremals for the functional
   \( v(x(t)) = \int_{0}^{t_f} \left( \frac{1}{4} \dot{x}^2(t) - t\dot{x}(t) - x(t) + \frac{1}{2} x(t)\dot{x}(t) \right) dt \)
   for the boundary conditions
   (a) \( x(0) = 1, \ t_f = 2, \) and \( x(2) = 10; \)
   (b) \( x(0) = 1, \ t_f = 10, \) and \( x(10) \) is free.

31. Find the equation of the curve that is an extremal for the functional
   \( v(x(t)) = \int_{0}^{t_f} \left( \frac{1}{2} \dot{x}^2(t) - t\dot{x}(t) + 2x^2(t) \right) dt \)
   subject to boundary conditions \( x(0) = 3/4, \ t_f = 1.0, \) and \( x(1) \) is free.

32. Given a dynamical system model
   \( \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \) and \( y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} x(t) \)
   and the associated performance index
   \( J = \int_{0}^{\infty} (y^2(t) + u^2(t)) dt \)
   (a) Assuming that \( u(t) = -Kx(t) \) is such that \( J \) is minimized, find the poles of the closed-loop systems.
   (b) Assume that \( x(0) = [1 \ 2]^T \), find the optimal value of \( J \).
   (c) Obtained the gain and phase margins of the closed loop system while the controller is designed based on LQR.

33. Show that if \( \lambda \) is an eigenvalue of the Hamiltonian matrix \( H \), then so is \(-\lambda\).

34. Given the Hamiltonian matrix corresponding to a dynamical system and its associated performance index. The Hamiltonian matrix is
   \( H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & -2 \end{bmatrix} \)
   and its eigenvalues are \( \lambda_1 = 3, \) and \( \lambda_2 = -3 \) respectively.
   (a) Find the solution to A.R.E using eigenvalue-eigenvector method.
   (b) Write the equation of the closed-loop system driven by the optimal controller.
35. For the standard infinite horizon (LQR) control problem, the optimal controller gain \((k)\) is given as
\[
k = [3.16 \quad 2.51],
\]
for the corresponding system and weighting matrices are described below:
\[
A = \begin{bmatrix} 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad R = 0.1
\]
(a) Obtain the Kalman’s return difference identity.
(b) Compute gain and phase margin of the closed loop system.

36. Consider a dynamical system model
\[
\dot{x}(t) = x(t) + u(t)
\]
and the cost function
\[
J = \frac{1}{2} x^2(10) + \frac{1}{2} \int_0^{10} (3x^2(t) + u^2(t)) dt
\]
do the following.
(a) Generate the Hamiltonian matrix for this control problem
(b) Generate the Riccati equation for this control problem
(c) Solve the optimal feedback gain.
(d) Obtain the steady state solution of the optimal gain.
(e) What is the pole of the closed-loop system at steady state gain?
(f) How does this closed-loop pole change if the state weighting matrix is increased by a factor of 10?

37. (a) Suppose that \(\lambda_i\) is an eigenvalue of the matrix \(A - BR^{-1}B^TP\) and \(v_i \neq 0\) is the corresponding eigenvector, where \(P\) is the symmetric positive definite solution of the A.R.E. Show that
\[
[v_i^T \quad (Pv_i)^T]^T
\]
is an eigenvector of the associated Hamiltonian matrix corresponding to the eigenvalue \(\lambda_i\).
(b) Show that if \(Q = Q^T > 0\), then the real parts of the eigen values of the matrix \(A - BR^{-1}B^TP\) are all negative, where \(P = P^T > 0\) is solution of the associated A.R.E.

38. Design a system for controlling the shaft angle of a dc motor. The entire state is measured and be used to generate the motor control voltage. A block diagram for the dc motor is shown in fig.5.

![Figure 5: A position control System](image-url)

Assume \(\theta(t)\) is the motor shaft angular position(angle)and \(U(t)\) is the dc voltage applied to the motor. Design a controller that minimizes the cost function.
\[ J(x(t), U(t)) = \frac{1}{2} \int_{0}^{0.4} \left( \theta^2(t) + 10^{-8}u^2(t) \right) dt \]

(a) Simulate the closed-loop system with an initial shaft position of \( \theta = 10 \) degree and an initial shaft velocity \( \dot{\theta} \) of zero.

(b) Using the steady state gains, simulate the closed-loop system with the same initial conditions.

(c) Plot the plant state and the control input for these two simulations and compare the results.

39. Given the following model of dynamic systems:

\[
\begin{align*}
(a) \quad \dot{x}_1(t) &= x_2(t) + 5 \\
\dot{x}_2(t) &= u(t) \\
(b) \quad \dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_2(t) + u(t)
\end{align*}
\]

where \( |u(t)| \leq 1 \). The performance index to be minimized is

\[ J = \int_{0}^{t_f} dt = t_f \]

Find the state feedback control law \( u = u(x_1, x_2) \) that minimizes \( J \) and drives the system from a given initial condition \( x(0) = [1, 2]^T \) to the final state \( x(t_f) = 0 \). Proceed as indicated below:

(i) Derive the equations of the optimal trajectories.

(ii) Derive the equation of the switching curve.

(iii) Write the expression for the optimal state-feedback controller.

(iv) Draw an implementable control scheme to realize such a controller in practice.

40. Consider a dynamic system

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 6 \\ -3 \end{bmatrix} \omega(t); \quad \text{and} \quad y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} x(t) + \eta(t) \]

Assume the state and input weighting matrices of \( LQR \) parameters are given as \( Q = 50I_{2\times2} \) and \( R = 5 \). The input and output noises are considered to be zero mean white noise \( \text{Gaussian} \) processes with known input and output noise covariances as \( \Omega = 0.3I_{2\times2} \) and \( V = 0.2 \) respectively. Design a controller \( u(t) = -K \hat{x}(t) \) based on \textit{Kalman filter}.

(a) Compute the \textit{controller gain} \( K \).

(b) Compute \textit{Kalman filter} gain \( L \).

(c) Compute \textit{gain and phase} margins of the system and draw the \textit{Nyquist} plot overlaid on the unit circle centered at the origin.

(d) Plot the response of the closed loop system with designed estimator based controller.

(e) Show the implementation scheme of \textit{Kalman filter} based controlled system.

(f) Design \textit{LQR} controller for the same system and compare the \textit{gain and phase} margins that are obtained in (c).
41. Consider a true plant
\[ G(s) = \frac{3e^{-0.1s}}{(2s + 1)(0.1s + 1)^2} \]

Obtain the additive uncertainty function (weight) \( \Delta_a(s) \) and sketch its magnitude for all frequencies \( \omega \geq 0 \), when the nominal model is \( G(s) = \frac{3}{(2s + 1)} \).

42. The nominal plant of a unity feedback control system is described as \( G_m(s) = \frac{1}{(s+2)(s+5)} \) and the controller \( K(s) = \frac{k}{s} \) is placed in the forward path. Using small-gain theorem, find the maximum value of \( k \) for which closed-loop system will be robustly stable while the bound of the multiplicative model uncertainty is \( \Delta_m(s) = \frac{0.25(1+4s)}{(1+0.25s)} \).

43. Consider the system shown in Fig.6 and assume the actual plant is given by
\[
G_m(s) = \frac{1}{s^2} \\
H(s) = \frac{2(s+1)}{s^2 + s + 1} \\
\Delta_m(s)
\]

Figure 6: An uncertain SISO System

\[ G(s) = \frac{2(s + 2)}{s^2(s^2 + s + 1)} \]

(a) Find a multiplicative uncertainty model for the system.
(b) Find \( M(s) \), the transfer function as seen by the multiplicative uncertainty.
(c) Determine if the closed-loop system is robustly stable under the multiplicative uncertainty (computed in (i).) using small-gain theorem

44. Consider the system shown in Fig.7.

(a) Draw a block diagram of the plant in standard form of a general system \( P(s) \) with uncertainties \( (\Delta = \text{diag.}[\delta_1 \ \delta_2]) \).
(b) Convert \( N - \Delta \) structure (i.e. uncertain feedback system) where \( \Delta = \text{diag.}[\delta_1 \ \delta_2] \) and \( N(s) = F(P(s), K(s)) \).
45. Consider the following nominal system stable (i.e without uncertainties):

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 + \Delta_1 & -9 + \Delta_2 & -3 + \Delta_3
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u(t);
\]

where \(\Delta_1, \Delta_2, \text{and} \Delta_3\) are uncertainties. Decompose the system as the nominal system \((A, B)\) and uncertainty \(\Delta\) as shown in Figure 8. Use the Small-Gain theorem to find the bound on the uncertainty \(\Delta\) that guarantees the stability of the system.

![Figure 8: An uncertain System](image)

46. For the system

\[
\dot{X}(t) = \begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix} X + \begin{bmatrix}
0 \\
1
\end{bmatrix} \text{ and } y = X
\]

(a) Calculate the \(H_2\)-norm of the transfer function \(G(s)\) of the above system using the method based on the Lyapunov equation.

(b) Evaluate of \(H_2\)-norm of \(G(s)\) using Cauchy residue theorem.

(c) Calculate \(H_\infty\) norm of \(G(s)\) by hand computation.

47. For the system

\[
G(s) = \begin{bmatrix}
\frac{1}{(s+2)^2} & \frac{1}{(s+1)(s+2)} \\
\frac{1}{s+2} & \frac{2}{(s+2)^2}
\end{bmatrix}
\]

Compute and plot (i) the maximum (\(\overline{\sigma}\)) (ii) minimum (\(\underline{\sigma}\)) singular values (SVD) of the T.F matrix \(G(j\omega)\) for \(0.1 \leq \omega \leq 100\) and hence obtain \(\|G(j\omega)\|_\infty\). Calculate the \(\|G(j\omega)\|_\infty\) directly using the MatLab command. Compare the result obtained in previous step.

48. For the system

\[
\dot{x}(t) = \begin{bmatrix}
-6.0 & -9.0 \\
1.0 & 0.0
\end{bmatrix} x(t) + \begin{bmatrix}
1 \\
0
\end{bmatrix} u(t); \quad y(t) = \begin{bmatrix}
0.0 & 20.0
\end{bmatrix} x(t)
\]

Compute (i)\(\|G(s)\|_2\) (open-loop transfer function) (ii) \(\|G(s)\|_\infty\) and interpret their significance.

49. A standard \(H_\infty\) control problem with no uncertainties in system matrices are given below:

\[
A = \begin{bmatrix}
0 & 1.0 \\
-6.0 & -5.0
\end{bmatrix} ; \quad B_1 = \begin{bmatrix}
1.0 \\
1.0
\end{bmatrix} ; \quad B_2 = \begin{bmatrix}
0 \\
1.0
\end{bmatrix} ;
\]

\[
C_1 = \begin{bmatrix}
1.0 & 1.0
\end{bmatrix} ; \quad C_2 = \begin{bmatrix}
1.0 & 0.0
\end{bmatrix} ; \quad D_{11} = 0.0; \quad D_{22} = 0.0, \quad D_{12} = 1.0, \text{ and } D_{21} = 2.0
\]
• Find the *four-block* representation of the generalized plant matrix $P(s)$.
• Find T.F from $\omega$ to $z$ in terms of controller gain matrix $k(s)$ with $u(s) = k(s)y(s)$.

50. Solve the $H_\infty$ control problem for the *generalized* plant $P(s)$ which is described in state-space form:

$$P(s) = \begin{bmatrix}
A & B_1 & B_2 \\
\cdot & \cdot & \cdot & \cdot \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22} \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0.2 \\
-1 & 1 & 0 \\
\end{bmatrix}$$

with controller $u(s) = k(s)y(s)$;

• Draw the generic $H_\infty$ block diagram of the system (indicating all internal and external signals).
• Verify that all rank conditions are met.
• Compute, $X_\infty, Y_\infty, Z_\infty, K_c$, and $K_e$ for $\gamma = 1.0$.
• Find the compensator T.F and the closed-loop poles.
• Draw the schematic diagram showing structure of the $H_\infty$ control system in time domain.