Lecture #3B: Problem solving session-I
Problem # 1: Consider a linear block code, C with parity check matrix given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

What is \((n, k)\) of C?

\[Solutions: \] Rank of \(H\) matrix is 3. So, \(n = 7, k = 7 - 3 = 4\).
Problem # 2: Consider the following binary block code, $C$,

$$C = \{000000, 110011, 011101, 111111\}$$

Is $C$ a linear block code? Justify your answer.

Solutions: No.
Linear block code

- **Problem # 2:** Consider the following binary block code, C,

\[ C = \{000000, 110011, 011101, 111111\} \]

Is C a linear block code? Justify your answer.

- **Solutions:** No.
- Sum of two codewords for a linear block code is a valid codeword.

Let \( v_0 = 000000, v_1 = 110011, v_2 = 011101, \) and \( v_3 = 111111, \) then \( v_1 + v_2, v_1 + v_3, v_2 + v_3, \) and \( v_1 + v_2 + v_3 \) must also be a valid codeword.

\[
\begin{align*}
    v_1 + v_2 &= 101110 \\
    v_1 + v_3 &= 001100 \\
    v_2 + v_3 &= 100010 \\
    v_1 + v_2 + v_3 &= 010001
\end{align*}
\]
Problem # 2 (contd.)

Thus a linear block code should have the following codewords

\[ C = \{000000, 110011, 011101, 111111, 101110, 001100, 100010, 010001\} \]

This is a (6,3) linear binary code.
Problem # 2 (contd.)

- Thus a linear block code should have the following codewords

\[ C = \{000000, 110011, 011101, 111111, 101110, 001100, 100010, 010001\} \]

- This is a (6,3) linear binary code.

- One example of generator matrix for this code

\[
G = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Problem # 2 (contd.): Generator matrix in systematic form

- How to write the generator matrix in systematic form?

\[
G = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Problem # 2 (contd.): Generator matrix in systematic form

- How to write the generator matrix in systematic form?

\[
G = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

- Row 3 → Row 3 + Row 1

\[
G = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

- Row 2 → Row 3 + Row 2

\[
G = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

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An introduction to coding theory
Problem # 2 (contd.): Generator matrix in systematic form

- Row 1 → Row 1 + Row 2

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Similarly parity check matrix in systematic form can be written as

\[
H = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Problem # 3: Let $H$ be the parity check matrix of an $(n,k)$ linear code $C$ that has both odd and even-weight codewords. Construct a new linear code $C_1$ with the following parity-check matrix

$$H_1 = \begin{bmatrix}
0 & 0 & \cdots & H \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
1 & 1 & \cdots & 1
\end{bmatrix}$$

Show that $C_1$ is an $(n+1, k)$ linear code.
Problem # 3: Let $H$ be the parity check matrix of an $(n,k)$ linear code $C$ that has both odd and even-weight codewords. Construct a new linear code $C_1$ with the following parity-check matrix

$$H_1 = \begin{bmatrix} 0 & | & H \\ 0 & | & 0 \\ \vdots & | & \vdots \\ 0 & | & \cdots \\ 1 & | & 1 \cdots 1 \end{bmatrix}$$

1. Show that $C_1$ is an $(n+1,k)$ linear code.
2. Show that every codeword of $C_1$ has even weight.
3. Show that $C_1$ can be obtained from $C$ by adding an extra parity check digit, denoted by $v_\infty$ to the left of each codeword $v$ as follows
**Problem #3:** Let $H$ be the parity check matrix of an $(n,k)$ linear code $C$ that has both odd and even-weight codewords. Construct a new linear code $C_1$ with the following parity-check matrix

$$H_1 = \begin{bmatrix} 0 & & & & H \\ \vdots & 0 & & & \vdots \\ 0 & & \ddots & & \vdots \\ \vdots & 0 & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & \end{bmatrix}$$

Show that $C_1$ is an $(n+1,k)$ linear code.

Show that every codeword of $C_1$ has even weight.

Show that $C_1$ can be obtained from $C$ by adding an extra parity check digit, denoted by $v_\infty$ to the left of each codeword $v$ as follows

1. if $v$ has odd weight, then $v_\infty = 1$, and
2. if $v$ has even weight, then $v_\infty = 0$
The matrix $H_1$ is an $(n - k + 1) \times (n + 1)$ matrix.

First we note that the $n - k$ rows of $H$ are linearly independent. It is clear that the first $(n - k)$ rows of $H_1$ are also linearly independent.
Problem # 3 (contd.)

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First we note that the $n - k$ rows of $H$ are linearly independent. It is clear that the first $(n - k)$ rows of $H_1$ are also linearly independent.

The last row of $H_1$ has a “1” at its first position but other rows of $H_1$ have a “0” at their first position. Any linear combination including the last row of $H_1$ will never yield a zero vector.

Thus all the rows of $H_1$ are linearly independent. Hence the row space of $H_1$ has dimension $n-k+1$. 
Problem # 3 (contd.)

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- The last row of $H_1$ has a “1” at its first position but other rows of $H_1$ have a “0” at their first position. Any linear combination including the last row of $H_1$ will never yield a zero vector.
- Thus all the rows of $H_1$ are linearly independent. Hence the row space of $H_1$ has dimension $n-k+1$.
- The dimension of its null space, $C_1$, is then equal to

$$\dim(C_1) = (n + 1) - (n - k + 1) = k$$

Hence $C_1$ is an $(n + 1, k)$ linear code.
Problem # 3 (contd.)

Show that every codeword of $C_1$ has even weight.

Solution: The last row of $H_1$ is an all-one vector.
Show that every codeword of $C_1$ has even weight.

**Solution:** The last row of $H_1$ is an all-one vector.

The inner product of a vector with odd weight and the all-one vector is “1”. Hence, for any odd weight vector $v$,

$$vH_1^T \neq 0$$

and $v$ cannot be a code word in $C_1$.

Therefore, $C_1$ consists of only even-weight code words.
Problem # 3 (contd.)

Show that $C_1$ can be obtained from $C$ by adding an extra parity check digit, denoted by $v_\infty$ to the left of each codeword $v$ as follows

1) if $v$ has odd weight, then $v_\infty = 1$, and
Show that $C_1$ can be obtained from $C$ by adding an extra parity check digit, denoted by $v_\infty$ to the left of each codeword $v$ as follows

1) if $v$ has odd weight, then $v_\infty = 1$, and
2) if $v$ has even weight, then $v_\infty = 0$.

**Solution:** Let $v$ be a code word in $C$. Then $vH^T = 0$. Extend $v$ by adding a digit $v_\infty$ to its left.
Problem # 3 (contd.)

Show that \( C_1 \) can be obtained from \( C \) by adding an extra parity check digit, denoted by \( v_∞ \) to the left of each codeword \( v \) as follows

1) if \( v \) has odd weight, then \( v_∞ = 1 \), and
2) if \( v \) has even weight, then \( v_∞ = 0 \)

Solution: Let \( v \) be a code word in \( C \). Then \( vH^T = 0 \). Extend \( v \) by adding a digit \( v_∞ \) to its left.

This results in a vector of \( n+1 \) digits,

\[ v_1 = (v_∞, v) = (v_∞, v_0, v_1, \cdots, v_{n-1}) \]

For \( v_1 \) to be a vector in \( C_1 \), we must require that

\[ v_1H_1^T = 0 \]
Problem # 3 (contd.)

- Note that the inner product of $v_1$ with any of the first $n-k$ rows of $H_1$ is 0.

- The inner product of $v_1$ with the last row of $H_1$ is
  \[ v_\infty + v_0 + v_1 + \cdots + v_{n-1} \]
Note that the inner product of $v_1$ with any of the first $n-k$ rows of $H_1$ is 0.

The inner product of $v_1$ with the last row of $H_1$ is

$$v_\infty + v_0 + v_1 + \cdots + v_{n-1}$$

For this sum to be zero, we must require that $v_\infty = 1$ if the vector $v$ has odd weight and $v_\infty = 0$ if the vector $v$ has even weight.

Therefore, any vector $v_1$ formed as above is a codeword in $C_1$, there are $2^k$ such codewords.
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$$v_\infty + v_0 + v_1 + \cdots + v_{n-1}$$

For this sum to be zero, we must require that $v_\infty = 1$ if the vector $v$ has odd weight and $v_\infty = 0$ if the vector $v$ has even weight.

Therefore, any vector $v_1$ formed as above is a codeword in $C_1$, there are $2^k$ such codewords.

The dimension of $C_1$ is $k$, these $2^k$ codewords are all the code words of $C_1$. 

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