Lecture #18: Convergence of turbo codes
Outline of the lecture

- Introduction
- Measures for convergence analysis of turbo codes
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  - El Gamal’s method
  - Density evolution
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  - EXIT charts

Transfer Characteristics of the turbo decoder
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  - Density evolution
  - EXIT charts
- Transfer Characteristics of the turbo decoder
- Threshold Calculation

Limitations of the threshold calculation methods.
Convergence analysis is used to explain the performance of the turbo code in the waterfall region.

For turbo iterative decoding, the extrinsic information from one decoder is fed as a-priori information to the other decoder.
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Initially, the decoder has no a-priori information about the information bits.

With increasing iterations, only input to the decoder that is changing is the a-priori information.
Basic idea of convergence analysis is to track how extrinsics information evolve with increasing iterations.

For a given $E_b/N_0$, convergence analysis methods relate a parameter related to the extrinsic information of the turbo decoder to a parameter related to the a-priori information of the turbo decoder.
Basic idea of convergence analysis is to track how extrinsics information evolve with increasing iterations.

For a given $E_b/N_0$, convergence analysis methods relate a parameter related to the extrinsic information of the turbo decoder to a parameter related to the a-priori information of the turbo decoder.

For asymptotically large block lengths, the smallest channel SNR for which iterative decoding converges is known as the \textit{iterative decoding threshold}.
Introduction

- Basic idea of convergence analysis is to track how extrinsics information evolve with increasing iterations.
- For a given $E_b/N_0$, convergence analysis methods relate a parameter related to the extrinsic information of the turbo decoder to a parameter related to the a-priori information of the turbo decoder.
- For asymptotically large block lengths, the smallest channel SNR for which iterative decoding converges is known as the **iterative decoding threshold**.
- Convergence analysis methods provide a tool to compute convergence thresholds for concatenated coding schemes using iterative decoding.
- They also help in the selection of constituent codes and puncturing patterns for turbo coding schemes for good performance in the waterfall region.

Adrish Banerjee  
Department of Electrical Engineering  
Indian Institute of Technology Kanpur  
Kanpur, Uttar Pradesh India

An introduction to coding theory

Measures for convergence analysis of turbo codes

- SNR [1].
- Density evolution [2].
- EXIT charts [3].

El-Gamal’s approach

- This method is based on Gaussian approximation of the output extrinsic information.

- The Gaussian approximation allows characterization of the turbo decoder by its SNR.
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- The Gaussian approximation allows characterization of the turbo decoder by its SNR.
- For an AWGN channel,

\[ z = x + n \]

where \( z \) is the received channel value, \( x \) is the transmitted bit (\( = \pm 1 \)), and \( n \) is Gaussian distributed with zero mean and variance \( N_0/2 \).

The log-likelihood or L-values are calculated as:

\[ Z = \ln \frac{p(z|x = +1)}{p(z|x = -1)} \quad A = \ln \frac{p(u = +1)}{p(u = -1)}, \]

where \( u(= \pm 1) \) represents an information bit.
El-Gamal’s approach

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where \( u(\pm 1) \) represents an information bit.

- For large block sizes, the probability distribution of the a-priori L-values \( p_A \), are assumed to be Gaussian. In particular, the a-priori L-value \( A \) can be modeled as

\[ A = \mu_A \cdot u + n_A \]

where the \( n_A \) is a zero mean Gaussian random variable with variance \( \sigma^2_A \) that satisfies the following condition

\[ \mu_A = \frac{\sigma^2_A}{2}. \]  

(consistency condition)

El-Gamal’s approach

- The a-priori information at the input of the decoder can be characterized by input SNR,

\[ \text{SNR}_i = \frac{\mu_A^2}{\sigma^2_A} = \frac{\mu_A}{2}. \]
El-Gamal’s approach

• The a-priori information at the input of the decoder can be characterized by input SNR,

\[ \text{SNR}_i = \frac{\mu_A^2}{\sigma_A^2} = \frac{\mu_A}{2} \]

• The extrinsic information at the output of the decoder can be characterized by output SNR calculated as follows

\[ P_e = Q(\sqrt{2\text{SNR}_0}) \]

where \( P_e \) is the bit error probability of the extrinsic information at the output of SISO decoder.

Viewing \( \text{SNR}_0 \) as a function of \( E_b/N_0 \), and \( \text{SNR}_i \), the transfer characteristics of the decoder can be written as,

\[ \text{SNR}_0 = T(\text{SNR}_i, E_b/N_0) \]
Transfer characteristics of a SISO decoder

**Step 1**: For a given SNR $E_b/N_0$, the distribution of a-priori L-value is generated for a particular mean $\mu_A$, and transmitted bits $u$.

**Step 2**: A SISO MAP decoder module is simulated. The inputs to the SISO module are the channel L-values, and the a-priori L-value generated in Step 1.
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Step 3: The mean, $\mu_E$, of the extrinsic information generated in step 2 is calculated.

Step 4: The mean $\mu_A$ is varied from zero to a large number and the steps 1-3 are repeated.

Adrish Banerjee
Department of Electrical Engineering
Indian Institute of Technology Kanpur, Kanpur, Uttar Pradesh India

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Transfer characteristics of a SISO decoder

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**Step 4**: The mean $\mu_A$ is varied from zero to a large number and the steps 1-3 are repeated.

**Step 5**: The set of $(SNR_i, SNR_0)$ for different values of $\mu_a$ is plotted. This is then used as the transfer characteristics for the SISO module for that particular code, and channel SNR $E_b/N_0$.
Threshold Calculation

Step 1: For a particular $E_b/N_0$, plot the transfer characteristics of SISO decoder for two constituent encoders on a reverse set of axes.

Step 2: If a tunnel exists, the channel SNR $E_b/N_0$ is reduced until the transfer characteristics touch or cross each other.
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Step 3 : If the transfer characteristics touch or cross each other, the channel SNR $E_b/N_0$ is increased until a tunnel exists.

Step 4 : The smallest channel SNR $E_b/N_0$ for which the transfer characteristics do not touch and a tunnel exists is the convergence threshold for that particular code.
Threshold Calculation

Rate $R=1/3$ turbo code

$E_b/N_0 = -0.2 \text{ dB}$

$[1 + D + D^3 + D^7 + D^8]$ $/ 1 + D + D^2 + D^3$

Decoder 1
Decoder 2

SNR/$SNR_0$

Adrish Banerjee
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh India
An introduction to coding theory
Decoding trajectory of a Turbo decoder

Step 1: For a particular $E_b/N_0$, plot the transfer characteristics of SISO decoder for two constituent encoders on a reverse set of axes.

Step 2: For a given $E_b/N_0$, initially SNR$_i = 0$ corresponding to the first iteration of decoder 1, we determine the resulting SNR$_0$ (vertically) using the transfer characteristics for decoder 1.
Decoding trajectory of a Turbo decoder

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Step 2: For a given $E_b/N_0$, initially $\text{SNR}_i = 0$ corresponding to the first iteration of decoder 1, we determine the resulting $\text{SNR}_0$ (vertically) using the transfer characteristics for decoder 1.

Step 3: Since the extrinsic information at the output of decoder 1 becomes the a-priori information at the input of decoder 2, the value of $\text{SNR}_0$ from decoder 1 becomes $\text{SNR}_i$ for the first iteration of decoder 2, and the resulting $\text{SNR}_0$ for decoder 2 is determined (horizontally) using the transfer characteristics for decoder 2.

Step 4: This procedure is repeated to trace the trajectory of iterative decoding.
Decoding trajectory of a Turbo decoder

**Step 1**: For a particular $E_b/N_0$, plot the transfer characteristics of SISO decoder for two constituent encoders on a reverse set of axes.

**Step 2**: For a given $E_b/N_0$, initially $\text{SNR}_i = 0$ corresponding to the first iteration of decoder 1, we determine the resulting $\text{SNR}_0$ (vertically) using the transfer characteristics for decoder 1.

**Step 3**: Since the extrinsic information at the output of decoder 1 becomes the a-priori information at the input of decoder 2, the value of $\text{SNR}_0$ from decoder 1 becomes $\text{SNR}_i$ for the first iteration of decoder 2, and the resulting $\text{SNR}_0$ for decoder 2 is determined (horizontally) using the transfer characteristics for decoder 2.

**Step 4**: This procedure is repeated to trace the trajectory of iterative decoding.

**Step 5**: If a tunnel exists between the two transfer characteristics, iterative decoding converges.
Density Evolution

- This method is based on tracking the actual densities of the extrinsic information during each half iteration.

- Generate input a-priori distribution based on the observed extrinsic information distribution.
Density Evolution

- This method is based on tracking the actual densities of the extrinsic information during each half iteration.
- Generate input a-priori distribution based on the observed extrinsic information distribution.
- Simulate SISO decoder, and from the generated extrinsic information find the distribution of the extrinsic information.

The a-priori/extrinsic information can be characterized by the $\text{SNR}_i/\text{SNR}_0$, where mean and variance of the a-priori/extrinsic information is calculated empirically.
Mutual Information is used to describe the flow of extrinsic information through soft in/soft out decoders.

The information content of the a-priori probabilities is measured by the mutual information $I_A = I(U; A)$ between the information bits $U$ and the a-priori $L$-values $A$. 
Mutual Information is used to describe the flow of extrinsic information through soft in/soft out decoders.

The information content of the a-priori probabilities is measured by the mutual information $I_A = I(U; A)$ between the information bits $U$ and the a-priori $L$-values $A$.

The input mutual Information $I(U; A)$ is calculated as:

$$I(U; A) \triangleq \frac{1}{2} \sum_{U = -1, 1} \int_{-\infty}^{\infty} p_A(\xi | U = u) \log \frac{p_A(\xi | U = u)}{p_A(\xi)} d\xi$$

The information content of the extrinsic a-posteriori probabilities is measured by the mutual information $I_E = I(U; E)$ between the information bits $U$ and the extrinsic $L$-values $E$. 

$$I(U; E) \triangleq \frac{1}{2} \sum_{U = -1, 1} \int_{-\infty}^{\infty} p_A(\xi | U = u) \log \frac{p_A(\xi | U = u)}{p_A(\xi)} d\xi$$
The information content of the extrinsic a-posteriori probabilities is measured by the mutual information $I_E = I(U; E)$ between the information bits $U$ and the extrinsic L-values $E$.

The probability distribution of the extrinsic L-values $p_E$, is computed experimentally from Monte Carlo simulations. $p_E$ is then used to calculate the output mutual information $I(U; E)$.

$$I(U; E) \triangleq \frac{1}{2} \sum_{U=-1,1} \int_{-\infty}^{\infty} p_E(\xi | U = u) \log \frac{p_E(\xi | U = u)}{p_E(\xi)} d\xi$$

Viewing $I_E$ as a function of $I_A$ and $E_b/N_0$, the extrinsic information transfer characteristic of an encoder is defined as

$$I_E = T(I_A, E_b/N_0).$$
Extrinsic Information Transfer Charts

An EXIT chart for a particular channel SNR $E_b/N_0$ can be formed by plotting the transfer characteristics of the two constituent encoders on reverse axes.

The EXIT chart can then be used to trace the trajectory of iterative decoding and to determine the convergence behavior of the constituent decoders.
Extrinsic Information Transfer Charts

- An EXIT chart for a particular channel SNR $E_b/N_0$ can be formed by plotting the transfer characteristics of the two constituent encoders on reverse axes.
- The EXIT chart can then be used to trace the trajectory of iterative decoding and to determine the convergence behavior of the constituent decoders.
- The existence of a “tunnel” implies convergence of iterative decoding.

As the channel SNR $E_b/N_0$ is lowered, the two transfer characteristics come closer together (the “tunnel” narrows) until the two curves meet.
Limitations of the convergence analysis methods

- Convergence analysis is based on asymptotically large block sizes. Practical systems use a finite block sizes.

- Some of the methods are based on Gaussian approximation of the distribution of the extrinsic information. This approximation may not hold.