An introduction to coding theory

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Feb. 27, 2017

Lecture #14A: Decoding of low density parity check codes-I
Outline of the talk

- Decoding on BSC: Bit Flipping Algorithm
  - Example 1: One transmission error case.
Outline of the talk

- Decoding on BSC: Bit Flipping Algorithm
  - Example 1: One transmission error case.
  - Example 2: Two transmission errors case.

Low-density parity check codes

```
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

- Example of a low density code matrix; n=20, j=3, k=4
Definitions

• The set of bits contained in a parity-check equation constitutes a **parity check set**.

• **Parity check set tree** is a representation of parity check set in a tree structure.
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  - An arbitrary bit $d$ is represented by the node of the base of the tree.
  - Each line rising from this node represents one of the parity-check sets containing $d$. 
Definitions

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- **Parity check set tree** is a representation of parity check set in a tree structure:
  - An arbitrary bit $d$ is represented by the node of the base of the tree.
  - Each line rising from this node represents one of the parity-check sets containing $d$.
  - The other nodes bits in these parity-check sets are represented by the nodes on the first tier of the tree.

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Definitions

- The set of bits contained in a parity-check equation constitutes a **parity check set**.
- **Parity check set tree** is a representation of parity check set in a tree structure.
  - An arbitrary bit \( d \) is represented by the node of the base of the tree.
  - Each line rising from this node represents one of the parity-check sets containing \( d \).
  - The other nodes' bits in these parity-check sets are represented by the nodes on the first tier of the tree.
  - The lines rising from tier 1 to tier 2 of the tree represent the other parity-check sets containing the bits on tier 1.
  - The nodes on tier 2 represent the other bits in those parity-check sets.

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Low-density parity check codes

<table>
<thead>
<tr>
<th>1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0</th>
<th>1 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0</td>
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- Example of a low density code matrix; \( n=20, j=3, k=4 \)
Parity-check set

<table>
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<tr>
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<tbody>
<tr>
<td>1</td>
<td>{1,2,3,4}</td>
</tr>
<tr>
<td>2</td>
<td>{5,6,7,8}</td>
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<tr>
<td>3</td>
<td>{9,10,11,12}</td>
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<tr>
<td>4</td>
<td>{13,14,15,16}</td>
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<tr>
<td>5</td>
<td>{17,18,19,20}</td>
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<tr>
<td>6</td>
<td>{1,5,9,13}</td>
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<tr>
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<td>{8,12,16,20}</td>
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<tr>
<td>12</td>
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<tr>
<td>13</td>
<td>{3,8,13,19}</td>
</tr>
<tr>
<td>14</td>
<td>{4,9,14,17}</td>
</tr>
<tr>
<td>15</td>
<td>{5,10,15,20}</td>
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</tbody>
</table>

Parity-check set tree
Example 1: Single transmission error case

Transmitted bits = \{0,0,0,0,0,1,1,1,1,1,1,0,1,0,1,1,0,0\}
Received bits = \{1,0,0,0,0,1,1,1,1,1,1,0,1,0,1,1,0,0\}

- The first bit is received in error.

Decoder will try to correct the error.
Decoding on BSC: Bit-Flipping Algorithm

- **Step 1:** Represent the code using parity check set tree.

Parity-check set

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Decoding on BSC: Bit-Flipping Algorithm

- **Step 2:** First Iteration: Check the parity-check sets containing bit # 1 to see if they are satisfied.

- All the three parity check set #1, 6, 11 are violated.
Decoding on BSC: Bit-Flipping Algorithm

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- All the three parity check set #1, 6, 11 are violated.
- Since all three of the parity check-set containing bit # 1 are violated, there is a strong possibility that bit #1 is in error.

Flip the first received bit # 1 from 1 to 0 and recompute the syndrome(check whether the parity constraints are satisfied).
Decoding on BSC: Bit-Flipping Algorithm

- **Step 2:** First Iteration: Check the parity-check sets containing bit #1 to see if they are satisfied.
  - All the three parity check set #1, 6, 11 are violated.
  - Since all three of the parity check-set containing bit #1 are violated, there is a strong possibility that bit #1 is in error.
  - Flip the first received bit #1 from 1 to 0 and recompute the syndrome (check whether the parity constraints are satisfied).
  - All the parity equations containing bit 1 are satisfied, hence the first bit is decoded as 0.

Example 2: Two transmission errors case

Transmitted bits = \{0,0,0,0,0,1,1,1,1,1,1,1,0,0,1,1,0,0\}
Received bits = \{1,1,0,0,0,0,1,1,1,1,1,1,0,1,0,1,1,0\}
- First two bits are received in error.
Example 2: Two transmission errors case

Transmitted bits = \{0,0,0,0,0,1,1,1,1,1,1,0,1,0,1,1,0,0\}
Received bits = \{1,1,0,0,0,0,1,1,1,1,1,1,1,0,1,0,1,1,0,0\}

- First two bits are received in error.
- Decoder will try to correct bit #1 and 2.
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Decoding on BSC: Bit-Flipping Algorithm

**Step 2:** First Iteration: Check the parity-check set containing bit #1 to see if they are satisfied.
Decoding on BSC: Bit-Flipping Algorithm

**Step 2:** First Iteration: Check the parity-check set containing bit #1 to see if they are satisfied.

- Two of the three parity check set #6 and 11 are violated. Parity check set #1 is satisfied.
- Since majority of the parity-check set containing bit #1 are violated, there is a strong possibility that first bit is in error.
Decoding on BSC: Bit-Flipping Algorithm

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- Two of the three parity check set #6 and 11 are violated. Parity check set #1 is satisfied.
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- Flip the first received bit from 1 to 0 and recompute the syndrome (check whether the parity constraints are satisfied).

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Decoding on BSC: Bit-Flipping Algorithm

- **Step 2:** First Iteration: Check the parity-check set containing bit #1 to see if they are satisfied.
- Two of the three parity check set #6 and 11 are violated. Parity check set #1 is satisfied.
- Since majority of the parity-check set containing bit #1 are violated, there is a strong possibility that first bit is in error.
- Flip the first received bit from 1 to 0 and recompute the syndrome (check whether the parity constraints are satisfied.
- Parity check-set #6 and 11 are satisfied, but #1 failed.
- Hence the first iteration is not sufficient to correct the errors.

- **Step 3:** Second Iteration: Check the parity-check set containing bits in the first tier of the parity check-set tree to see if they are satisfied.
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- Parity check set containing bits 3(#8 and 13), 4(#9 and 14), 5(#2 and 15), 9(#3 and 14), 13(#4 and 13), 12(#3 and 10), 18(#5 and 8) are satisfied. One of the parity check set containing bit 6(#2) is also satisfied.

- Both the parity check set (#7 and 12) containing bit 2 and one of the parity check set containing (#7) containing bit 6 are violated.
Step 3: Second Iteration: Check the parity-check set containing bits in the first tier of the parity check-set tree to see if they are satisfied.

Parity check set containing bits 3(#8 and 13), 4(#9 and 14), 5(#2 and 15), 9(#3 and 14), 13(#4 and 13), 12(#3 and 10), 18(#5 and 8) are satisfied. One of the parity check set containing bit 6(#2) is also satisfied.

Both the parity check set (#7 and 12) containing bit 2 and one of the parity check set containing (#7) containing bit 6 are violated.

Bit 2 is common in all of these three parity check set as well as the parity check #1 which was violated after first iteration.

Hence, there is a strong possibility that second bit is in error.
Decoding on BSC: Bit-Flipping Algorithm

- Flip the second received bit #2 from 1 to 0 and recompute the syndrome (check whether the parity constraints are satisfied.

- All the parity check sets at first tier are satisfied.
Flip the second received bit #2 from 1 to 0 and recompute the syndrome (check whether the parity constraints are satisfied).

- All the parity check sets at first tier are satisfied.
- Now we check the parity-check sets at zero tier (containing first bit) and they are also satisfied.

Hence the first and second bits are decoded as 0's.
Decoding on BSC: Bit Flipping Algorithm

- The decoder computes all the parity checks and then changes any bit that is contained in more than some fixed number of unsatisfied parity-check equations.

- Using these new values, the parity checks are recomputed, and the process is repeated until the parity checks are all satisfied.
Decoding on BSC: Bit Flipping Algorithm

- The decoder computes all the parity checks and then changes any bit that is contained in more than some fixed number of unsatisfied parity-check equations.
- Using these new values, the parity checks are recomputed, and the process is repeated until the parity checks are all satisfied.
- If the parity check sets are small, this decoding procedure is reasonable, since most of the parity-check sets will contain either one transmission error or no transmission errors.
- Thus when most of the parity-check equation checking on a digit are unsatisfied, there is a strong indication that that digit is in error.