Lecture #12: Performance Bounds for Convolutional Codes
Introduction

- We will analyze the performance of maximum likelihood decoding for a convolutional code over binary symmetric channel.

Without loss of generality, we assume that the all zero codeword $0$ is transmitted from a $(3, 1, 2)$ nonsystematic feedforward encoder with

$$G(D) = [1 + D \quad 1 + D^2 \quad 1 + D + D^2]$$
We will analyze the performance of maximum likelihood decoding for a convolutional code over binary symmetric channel.

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\[ G(D) = [1 + D \ 1 + D^2 \ 1 + D + D^2] \]

The Input Output Weight Enumerating Function (IOWEF) of this encoder is given by

\[ A(W, X, L) = \frac{X^7 WL^3}{1 - XWL(1 + X^2 L)} \]

\[ = X^7 WL^3 + X^8 W^2 L^4 + X^9 W^3 L^5 + X^{10}(W^2 L^5 + W^4 L^6 + \cdots) \]

A first event error happens at an arbitrary time t if the all zero path is eliminated for the first time in favor of an incorrect path.
First Error Event

- The incorrect path must have diverged from all zero state at some time and has now remerged at time $t$ for the first time.

So, it must be one of the path enumerated by the codeword weight enumerating function.
First Error Event

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- So, it must be one of the path enumerated by the codeword weight enumerating function.
- Assuming that the incorrect path has weight $d$, a first event error happens with probability

$$P_d = \begin{cases} \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^e (1-p)^{d-e} & \text{odd} \\ \frac{1}{2} \binom{d}{d/2} p^{d/2} (1-p)^{d/2} + \sum_{e=\frac{d}{2}+1}^{d} \binom{d}{e} p^e (1-p)^{d-e} & \text{even} \end{cases}$$

Convolutional codes

- All incorrect paths of length $t$ branches or less can cause a first event error at time $t$. 
Convolutional codes

- All incorrect paths of length \( t \) branches or less can cause a first event error at time \( t \).
- Thus the first event error probability at time \( t \) can be bounded using union bound by the sum of the error probabilities of each of these paths.

If all incorrect paths of length greater than \( t \) are also included, then the first event error probability at any time \( t \) can be bounded by

\[
P_f(E) < \sum_{d=d_{\text{free}}}^{\infty} A_d P_d
\]

where \( A_d \) is the number of codewords of weight \( d \).
First Error Event

- For odd \( d \), we can write

\[
P_d = \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^e (1 - p)^{d-e}
\]

\[
< \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^{d/2} (1 - p)^{d/2}
\]

\[
= p^{d/2} (1 - p)^{d/2} \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^e (1 - p)^{d/2 - e}
\]

\[
< p^{d/2} (1 - p)^{d/2} \sum_{e=0}^{d} \binom{d}{e} = 2^d p^{d/2} (1 - p)^{d/2}
\]

- Similarly, for even \( d \), we have

\[
P_d = \frac{1}{2} \binom{d}{d/2} p^{d/2} (1 - p)^{d/2} + \sum_{e=(d/2)+1}^{d} \binom{d}{e} p^e (1 - p)^{d-e}
\]

\[
< \sum_{e=(d/2)}^{d} \binom{d}{e} p^e (1 - p)^{d-e}
\]

\[
< \sum_{e=(d/2)}^{d} \binom{d}{e} p^{d/2} (1 - p)^{d/2}
\]

\[
= p^{d/2} (1 - p)^{d/2} \sum_{e=(d/2)}^{d} \binom{d}{e}
\]

\[
< p^{d/2} (1 - p)^{d/2} \sum_{e=0}^{d} \binom{d}{e} = 2^d p^{d/2} (1 - p)^{d/2}
\]
Hence,

$$P_f(E) < \sum_{d=\text{free}}^{\infty} A_d[2\sqrt{p(1-p)}]^d$$

$$= A(X)|_{X=2\sqrt{p(1-p)}}$$

We have event error probability at time $t$ upper bounded by first event error probability, hence

$$P(E) < A(X)|_{X=2\sqrt{p(1-p)}}$$
Event error probability

- Hence,

\[ P_f(E) < \sum_{d=d_{\text{free}}}^{\infty} A_d [2 \sqrt{p(1-p)}]^d \]

\[ = A(X) \bigg|_{X=2 \sqrt{p(1-p)}} \]

- We have event error probability at time \( t \) upper bounded by first event error probability, hence

\[ P(E) < A(X) \bigg|_{X=2 \sqrt{p(1-p)}} \]

- For small \( p \), the bound is dominated by the first time, thus event error probability can be approximated as

\[ P(E) \approx A_{d_{\text{free}}} [2 \sqrt{p(1-p)}]^{d_{\text{free}}} \]

Bit error probability

- The bit error probability can be bounded by

\[ P_b(E) < \sum_{d=d_{\text{free}}}^{\infty} B_d P_d \]

where \( B_d \) is the total number of nonzero information bits on all weight-\( d \) paths, divided by the number of information bits \( k \) per unit time.
The bit error probability can be bounded by

$$P_b(E) < \sum_{d=d_{\text{free}}}^{\infty} B_d P_d$$

where $B_d$ is the total number of nonzero information bits on all weight-$d$ paths, divided by the number of information bits $k$ per unit time.

Then we can write

$$P_b(E) < \sum_{d=d_{\text{free}}}^{\infty} B_d [2\sqrt{p(1-p)}]^d = B(X)|_{X=2\sqrt{p(1-p)}}$$

Multiple error events

Multiple error events
Different error events configuration
Example: Computation of Event error probability

For the (3, 1, 2) encoder calculate the event error probability for crossover probability of \( p = 10^{-2} \) for binary symmetric channel.
Example: Computation of Event error probability

- For the (3, 1, 2) encoder calculate the event error probability for crossover probability of $p = 10^{-2}$ for binary symmetric channel.
- $d_{\text{free}} = 7$ and $A_{d_{\text{free}}} = 1$, then we have

$$P(E) \approx 2^7 p^{7/2} = 1.28 \times 10^{-5}$$

Example: Computation of bit error probability

- Calculate the bit error probability for the same encoder for $p = 10^{-2}$.
Example: Computation of bit error probability

- Calculate the bit error probability for the same encoder for $p = 10^{-2}$.
- The bit weight enumerating function is given by

$$B(X) = \frac{(1/k) \partial A(W, X)}{\partial W} \bigg|_{W=1}$$

$$= \frac{\partial [X^7W/(1 - XW - X^3W)]}{\partial W} \bigg|_{W=1} X^7$$

$$= \frac{X^7}{(1 - 2X + X^2 - 2X^3 + 2X^4 + X^6)}$$

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