Lecture #10: Decoding of convolutional codes-I: Viterbi algorithm
Convolutional codes

Outline of the lecture:
- Viterbi decoding of \((n, 1, m)\) code.
- Example: Viterbi decoding of \((2, 1, 2)\) convolutional code on BSC.
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- Since one information bit enters the encoder at each time instant, there are two branches leaving each state in the trellis diagram.
- For time, \(l > m\), there are also two branches merging into each state.
- The encoding of an information sequence is equivalent to tracing a path through a trellis.
Viterbi decoding of \((n, 1, m)\) code

- The encoder is returned to all zero sequence after an \(L\) bit information sequence,

\[
u = (u_0, u_1, u_2, \cdots, u_{L-1})
\]

During the termination process, the number of states are reduced by half until all trellis paths converge back to the all-zero state.
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- For \(l \leq m\), there is exactly one path of length \(l\), entering each node at level(time) \(l\).
- For \(l > m\), there are exactly \(2^{l-m}\) paths of length \(l\), entering each node at level(time) \(l\).
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- There are total \(2^l\) paths of length \(l\).

On BSC:
- Let the information sequence of length \(L\)

\[ u = (u_0, u_1, \cdots, u_l, \cdots, u_{L-1}) \]

is encoded into code sequence of length \(N \overset{\Delta}{=} (L + m)n\)

\[ v = (v_0, v_1, \cdots, v_l, \cdots, v_{L+m-1}) \]
Viterbi decoding of \((n, 1, m)\) code

On BSC:
- Let the information sequence of length \(L\)
  \[
  \mathbf{u} = (u_0, u_1, \cdots, u_L, \cdots, u_{L-1})
  \]
  is encoded into code sequence of length \(N \equiv (L + m)n\)
  \[
  \mathbf{v} = (v_0, v_1, \cdots, v_l, \cdots, v_{L+m-1})
  \]
- If the code sequence \(\mathbf{v}\) is transmitted over a channel, let the received sequence is,
  \[
  \mathbf{r} = (r_0, r_1, \cdots, r_L, \cdots, r_{L+m-1}),
  \]
  where the \(l^{th}\) received block is
  \[
  r_l = (r_1^{(1)}, r_1^{(2)}, \cdots, r_1^{(n)}).
  \]

A maximum likelihood decoder finds a path through the trellis that maximizes the path conditional probability

\[
P(\mathbf{r}|\mathbf{v}) = \prod_{l=0}^{L+m-1} P(\mathbf{r}_l|\mathbf{v}_l)
\]

where the branch conditional probability

\[
P(\mathbf{r}_l|\mathbf{v}_l) = \prod_{i=1}^{n} P(r_i^{(i)}|v_i^{(i)})
\]
Viterbi decoding of \((n, 1, m)\) code

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where the branch conditional probability

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P(r_l|v_l) = \prod_{l=1}^{n} P(r^{(i)}_l|v^{(i)}_l)
\]

- The bit conditional probabilities \(P(r^{(i)}_l|v^{(i)}_l)\) are the channel transition probabilities.

Maximizing \(P(r|v)\) is equivalent to maximizing

\[
M(r|v) \overset{\Delta}{=} \log P(r|v)
\]
Viterbi decoding of $(n, 1, m)$ code

Maximizing $P(r|v)$ is equivalent to maximizing

$$M(r|v) \triangleq \log P(r|v)$$

$M(r|v)$ is called the path metric.

$$M(r|v) = \sum_{l=0}^{L+m-1} \log P(r_l|v_l)$$

$$= \sum_{l=0}^{L+m-1} M(r_l|v_l), \quad \text{(branch metrics)}$$

$$M(r_l|v_l) = \sum_{i=1}^{n} \log P(r_l^{(i)}|v_l^{(i)})$$

$$= \sum_{i=1}^{n} M(r_l|v_l), \quad \text{(bit metrics)}$$

The partial path metric for the first $j$ branches of a path $v$ is given by

$$M([r|v]_j) = \sum_{l=0}^{j-1} M(r_l|v_l)$$
Viterbi decoding of \((n, 1, m)\) code

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  \[
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  \]

- For BSC, the maximum likelihood decoder decodes the received sequence \(r\) into code sequence \(v\) that minimizes the Hamming distance \(d(r, v)\)

The Viterbi algorithm is a computationally efficient method of finding the path through the trellis with the best metric.
Viterbi decoding of \((n, 1, m)\) code

- The Viterbi decoder proceeds through the trellis level by level in search of the path with the best metric.
- At each level, the decoder compares the metric of all partial paths entering each state.
Viterbi decoding of \((n, 1, m)\) code

- The Viterbi decoder proceeds through the trellis level by level in search of the path with the best metric.
- At each level, the decoder compares the metric of all partial paths entering each state.
- The decoder stores the partial path entering each state with the best metric (survivor path) and eliminates all other partial paths.

For \(m \leq l \leq L\), there are total \(2^m\) survivors.
Viterbi decoding of \((n, 1, m)\) code

- The Viterbi decoder proceeds through the trellis level by level in search of the path with the best metric.
- At each level, the decoder compares the metric of all partial paths entering each state.
- The decoder stores the partial path entering each state with the best metric (survivor path) and eliminates all other partial paths.
- For \(m \leq l \leq L\), there are total \(2^m\) survivors.
- The number of survivors decrease during the termination process, until at time \(l = L + m\) when there is only one survivor left.

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An introduction to coding theory
Viterbi decoding of \((n, 1, m)\) code

**Viterbi Algorithm:**

1. **Step 1:** Starting at level \(l = m\) in the trellis, compute the partial metric for the single path entering each \(m^{th}\) level state. Store the survivor path and its metric for each state.

2. **Step 2:** Increase time \(l\) by one. Compute the partial metric for all the paths entering at the \((l + 1)^{th}\) level state by adding the branch metric entering that state to the metric of the connecting survivor path at the previous \(l^{th}\) level state. Store the survivor path and its metric for each state.
Viterbi decoding of \((n, 1, m)\) code

Viterbi Algorithm:

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3. **Step 3:** Repeat Step 2 until you are at the end of the trellis \((l = L + m)\).

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Example:

This \((2, 1, 2)\) convolutional code with \(L = 5\) is used on a BSC. The received sequence is

\[ r = (01, 11, 10, 10, 00, 11, 10) \]
Trellis diagram of \((2, 1, 2)\) convolutional code with \(L = 5\).
Viterbi Decoding

\[ r = (01, 11) \]

\[ \mathbf{v}_1 = (00, 00) \quad d(\mathbf{v}_1, r) = 3 \]

\[ \mathbf{v}_2 = (00, 11) \quad d(\mathbf{v}_2, r) = 1 \]

\[ \mathbf{v}_3 = (11, 01) \quad d(\mathbf{v}_3, r) = 2 \]

\[ \mathbf{v}_4 = (11, 10) \quad d(\mathbf{v}_4, r) = 2 \]
Viterbi Decoding

\[ r = (01, 11 10 10) \]

Level 4

Viterbi Decoding

\[ r = (01, 11 10 10 00) \]

Level 5
Viterbi Decoding

Level 6

Level 7

**r** = (01, 11 10 10 00 11)

**r** = (01, 11 10 10 00 11 10)
Viterbi Decoding

$\hat{v} = (00, 11 10 10 00 01 11)$

Level 7

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