Lecture #9A: Convolutional codes: Classification, Realization
Outline of the lecture

- Classification of convolutional encoder
  - Feedforward encoder
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  - Feedforward encoder
  - Feedback encoder
- Equivalent encoder
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  - Feedforward encoder
  - Feedback encoder
- Equivalent encoder
- Catastrophic encoder

- Controller canonical form realization
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- Catastrophic encoder
- Controller canonical form realization
- Observer canonical form realization

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An introduction to coding theory
Classification of convolutional encoders

Feedforward Encoder:
- The encoder corresponding to a polynomial generator matrix does not contain any feedback path, and hence it is known as a feedforward encoder.

\[ u(D) \rightarrow \square \rightarrow v(D) \]

(b)(a)

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Feedforward Encoder:

- The encoder corresponding to a polynomial generator matrix does not contain any feedback path, and hence it is known as a feedforward encoder.
- The output of a feedforward encoder can be represented as a linear combination of the current input and a finite number of past inputs. This is also referred as nonrecursive encoder.
- In figure, the encoder diagram of a rate $R = 1$, 2-state feedforward encoder with generator matrix $G(D) = [1 + D]$ is shown using a shift register implementation.

![Encoder Diagram](image)

Example: Feedforward encoder
Classification of convolutional encoders

Feedback Encoder:

- The encoder corresponding to a rational generator matrix with at least one nonpolynomial transfer function contains a feedback path and is known as a feedback encoder.

The output of a feedback encoder can be represented as a linear combination of past inputs as well as past outputs.
Classification of convolutional encoders

Feedback Encoder:
- The encoder corresponding to a rational generator matrix with at least one nonpolynomial transfer function contains a feedback path and is known as a feedback encoder.
- The output of a feedback encoder can be represented as a linear combination of past inputs as well as past outputs.
- Hence the output depends on infinite number of past inputs. This is also sometimes referred as recursive encoder.

In figure, the encoder diagram of a rate $R = 1/2$, 2-state feedback encoder with generator matrix $G(D) = [1 \quad \frac{1}{1+D}]$ is shown.

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An introduction to coding theory
Classification of convolutional encoders

Systematic Encoder:

- A rate $R = k/n$ convolutional encoder whose $k$ information sequences appear unchanged among the $n$ code sequences is called a systematic encoder, and its generator matrix is called a systematic generator matrix.
Classification of convolutional encoders

Systematic Encoder:

- A rate $R = k/n$ convolutional encoder whose $k$ information sequences appear unchanged among the $n$ code sequences is called a systematic encoder, and its generator matrix is called a systematic generator matrix.
- In figure, a systematic rate $R = 1/2$ feedback convolutional encoder is shown.

\[ \begin{align*}
&u(D) \\
&\downarrow \\
&\oplus \\
&\downarrow \\
&v(D)
\end{align*} \]

\[ \begin{align*}
&0 \quad 0/00 \\
&1/10 \quad 1/11 \\
&0/00 \\
&1 \quad 0/01
\end{align*} \]

Nonsystematic Encoder:

- In a nonsystematic convolutional encoder, the $k$ information sequences do not appear unchanged in the $n$ code sequences.
Classification of convolutional encoders

Nonsystematic Encoder:
- In a nonsystematic convolutional encoder, the k information sequences do not appear unchanged in the n code sequences.
- In figure, a nonsystematic rate $R = 1/2$ feedforward convolutional encoder is shown.

![Diagram of a nonsystematic convolutional encoder]

Equivalent Encoder

- Two convolutional generator matrices $G(D)$ and $G'(D)$ are equivalent if they encode the same code.
Two convolutional generator matrices $G(D)$ and $G'(D)$ are equivalent if they encode the same code.

Two convolutional encoders are equivalent if their generator matrices are equivalent.

Two generator matrices $G(D)$ and $G'(D)$ are equivalent if and only if there exists a rational invertible matrix $T(D)$ such that

$$G'(D) = T(D)G(D)$$
Equivalent Encoder

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- Two convolutional encoders are equivalent if their generator matrices are equivalent.
- Two generator matrices $G(D)$ and $G'(D)$ are equivalent if and only if there exists a rational invertible matrix $T(D)$ such that
  \[ G'(D) = T(D)G(D) \]

- Example: The generator matrix, $G(D) = [1 \ 1 \ 1 + D]$ and $G'(D) = [1 + D \ 1]$ are equivalent.

\[
G(D) = \begin{bmatrix} 1 + D & D & 1 + D \\ D & 1 & 1 \end{bmatrix}
\]
Equivalent Encoder

- Consider the nonsystematic encoder
  \[ G(D) = \begin{bmatrix} 1 + D & D & 1 + D \\ D & 1 & 1 \end{bmatrix} \]

- Step 1: Row 1 \( \implies \) \([1/(1+D)][\text{Row 1}]\).
  \[ G_1(D) = \begin{bmatrix} 1 & D/(1 + D) & 1 \\ D & 1 & 1 \end{bmatrix} \]

- Step 2: Row 2 \( \implies \) Row 2 + [D][Row 1].
  \[ G_2(D) = \begin{bmatrix} 1 & D/(1 + D) \\ 0 & (1 + D + D^2)/(1 + D) & 1 + D \end{bmatrix} \]
Equivalent Encoder

- Consider the nonsystematic encoder

\[
G(D) = \begin{bmatrix}
1 + D & D & 1 + D \\
D & 1 & 1
\end{bmatrix}
\]

- Step 1: Row 1 \implies [1/(1+D)][Row 1].

\[
G_1(D) = \begin{bmatrix}
1 & D/(1 + D) & 1 \\
D & 1 & 1
\end{bmatrix}
\]

- Step 2: Row 2 \implies Row 2 + [D][Row 1].

\[
G_2(D) = \begin{bmatrix}
1 & D/(1 + D) & 1 \\
0 & (1 + D + D^2)/(1 + D) & 1 + D
\end{bmatrix}
\]

- Step 3: Row 2 \implies [(1+D)/(1 + D + D^2)][Row 2].

\[
G_3(D) = \begin{bmatrix}
1 & D/(1 + D) & 1 \\
0 & 1 & (1 + D^2)/(1 + D + D^2)
\end{bmatrix}
\]

Step 4: Row 1 \implies Row 1 + [D/(1+D)][Row 2].
**Equivalent Encoder**

- Step 4: Row 1 $\implies$ Row 1 + [D/(1+D)][Row 2].
- Modified Systematic generator matrix

$$G'(D) = \begin{bmatrix} 1 & 0 & 1/(1 + D + D^2) \\ 0 & 1 & (1 + D^2)/(1 + D + D^2) \end{bmatrix}$$

**Catastrophic Encoder**

- A convolutional encoder is catastrophic if it encodes some information sequence with infinitely many non-zero symbols into a code sequence with finitely many non-zero symbols.
A convolutional encoder is catastrophic if it encodes some information sequence with infinitely many non-zero symbols into a code sequence with finitely many non-zero symbols.

This means that a finite number of channel errors may result in infinitely many errors in the receiver.

Example:

\[ G(D) = [1 + D \quad 1 + D^2] \]
Catastrophic Encoder

- A convolutional encoder is catastrophic if it encodes some information sequence with infinitely many non-zero symbols into a code sequence with finitely many non-zero symbols.
- This means that a finite number of channel errors may result in infinitely many errors in the receiver.
- Example:

\[ G(D) = [1 + D, 1 + D^2] \]

- If the input sequence \( u(D) = \frac{1}{1 + D} = 1 + D + D^2 + \cdots \), then the output sequence, \( [1, 1 + D] \) has only weight 3, even though the information sequence has infinite weight.

Controller Canonical Form Realization

- In controller canonical form realization, to realize a rate \( R = k/n \) convolutional encoder, \( k \) shift register are used for input sequences, and \( n \) adders are used to form the output sequences.
Controller Canonical Form Realization

- In controller canonical form realization, to realize a rate $R = k/n$ convolutional encoder, $k$ shift register are used for input sequences, and $n$ adders are used to form the output sequences.
- The $k$ input sequences enter the shift registers at the left end of each shift register.
- The $n$ adders used to obtain output sequences are external to the shift register.
In controller canonical form realization, to realize a rate $R = k/n$ convolutional encoder, $k$ shift register are used for input sequences, and $n$ adders are used to form the output sequences.

- the $k$ input sequences enter the shift registers at the left end of each shift register.
- The $n$ adders used to obtain output sequences are external to the shift register.
- In Figure 3.6(a) (next page), a rate $R = 1$, nonsystematic convolutional encoder with following generator function $G(D)$ is implemented in controller canonical form realization.

$$G(D) = \left[ \frac{f_0 + f_1 D + \cdots + f_{m-1} D^{m-1} + f_mD^m}{1 + q_1 D + q_2 D^2 + \cdots + q_mD^m} \right]$$
Observer Canonical Form Realization

- In observer canonical form realization, to realize a rate $R = k/n$ convolutional encoder, $n$ shift register are used for output sequences.
- The $k$ input sequences enter the adders internal to the shift registers.
In observer canonical form realization, to realize a rate $R = k/n$ convolutional encoder, $n$ shift register are used for output sequences. The $k$ input sequences enter the adders internal to the shift registers. The lowest degree term in the generator polynomial represent the connections to the right hand side of the shift registers.

In Figure 3.6(b) (next page), a rate $R = 1$, nonsystematic convolutional encoder with following generator function $G(D)$ is implemented in observer canonical form realization.

$$G(D) = \left[ \frac{f_0 + f_1 D + \cdots + f_{m-1} D^{m-1} + f_m D^m}{1 + q_1 D + q_2 D^2 + \cdots + q_m D^m} \right]$$
Observer Canonical Form Realization

Figure 3.6

Realization of Convolutional encoder

Example 3.7:
- Let’s consider a rate $R = 2/3$ systematic feedforward encoder with generator matrix

$$G(D) = \begin{bmatrix} 1 & 0 & 1 + D + D^2 \\ 0 & 1 & 1 + D \end{bmatrix}. $$
Realization of Convolutional encoder

Example 3.7:

- Let’s consider a rate $R = 2/3$ systematic feedforward encoder with generator matrix

$$G(D) = \begin{bmatrix} 1 & 0 & 1 + D + D^2 \\ 0 & 1 & 1 + D \end{bmatrix}.$$  

- The parity check matrix can be written as

$$H(D) = \begin{bmatrix} h^{(0)}(D) & h^{(1)}(D) & 1 \end{bmatrix} = \begin{bmatrix} 1 + D + D^2 & 1 + D & 1 \end{bmatrix}.$$  

- The controller canonical form realization results in $(3, 2, 3)$ encoder.
Realization of Convolutional encoder

**Example 3.7:**
- Let’s consider a rate $R = 2/3$ systematic feedforward encoder with generator matrix

$$G(D) = \begin{bmatrix} 1 & 0 & 1 + D + D^2 \\ 0 & 1 & 1 + D \end{bmatrix}.$$  

- The parity check matrix can be written as

$$H(D) = \begin{bmatrix} h^{(0)}(D) & h^{(1)}(D) & 1 \end{bmatrix} = \begin{bmatrix} 1 + D + D^2 & 1 + D & 1 \end{bmatrix}.$$  

- The controller canonical form realization results in $(3, 2, 3)$ encoder.
- The observer canonical form realization results in $(3, 2, 2)$ encoder.

Controller Canonical Form Realization

![Diagram](example_3.7_diagram.png)
Minimal encoder

A generator matrix of a convolutional code is minimal if its number of states is minimal over all equivalent generator matrices.
A generator matrix of a convolutional code is minimal if its number of states is minimal over all equivalent generator matrices.

A minimal encoder is a realization of a minimal encoding matrix $G(D)$ with the minimal number of memory elements over all realizations of $G(D)$. 