Prof. Kulkarni
Lec - 5 - 21/10/09

LAST LECTURE

DYNAMIC SYSTEMS
- LINEAR $\dot{x} = ax + bu$
- NON-LINEAR $\dot{x} = g(x, u)$

RESPONSE OF LINEAR (TIME INVARIANT) SYSTEMS $\dot{x} = \underline{\underline{a(t)}} \cdot x$

$$\dot{x} = ax \rightarrow x(t) = e^{at}x(0)$$

$$x(t) = e^{a(t-t_0)}x(t_0)$$

$$\dot{x} = ax + bu \qquad ?$$

$$\boxed{x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} \cdot b\, u(\tau)\, d\tau}$$

$$\dot{x}(t) = a e^{at} x(0) + \frac{d}{dt}\left[e^{at} \int_0^t e^{-a\tau} b\, u(\tau)\, d\tau \right]$$

$$= \left\{ a e^{at} x(0) + a e^{at} \cancel{\frac{d}{dt}} \left[\int_0^t e^{-a\tau} b\, u(\tau)\, d\tau \right] \right\}$$

$$+ \; e^{at} \frac{d}{dt} \left[\int_0^t e^{-a\tau} b\, u(\tau)\, d\tau \right]$$

$$\therefore \ \dot{x}(t) = a\left[e^{at}x(0) + \int_0^t e^{a(t-\tau)} b\,u(\tau)\,d\tau\right] \quad - \mathrm{I}$$

$$+ \ e^{at} \cdot e^{-at} b\,u(t) \quad - \underline{\mathrm{II}}$$

$$= \ a\,x \ + \ b\,u.$$

<u>ALSO VERIFY THAT</u>

$$x(t)\Big|_{t=0} = e^{0\cdot t}x(0) + \int_0^0 e^{a(t-\tau)} b\,u(\tau)\,d\tau$$

$$= x(0)$$

RESPONSE OF HIGHER ORDER COUPLED SYSTEMS

$$\dot{x} = A x$$

eg:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\underline{\dot{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

TRANSFORMATIONS

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y}$$

$\therefore$

$$P\dot{y} = APy$$

$$\therefore \quad \dot{y} = \underbrace{P^{-1}AP}_{\begin{bmatrix} * & * \\ * & * \end{bmatrix}} y .$$

DYNAMICAL EQNS OF NEW VARIABLE

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{P^{-1}AP} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1 = e^{\lambda_1 t} y_1(0), \quad y_2 = e^{\lambda_2 t} y_2(0).$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} e^{\lambda_1 t} \left(\underbrace{\begin{bmatrix} q_{11} & q_{12} \end{bmatrix}}_{2\times 1} \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{1\times 2} \right)$$

$$+ \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} e^{\lambda_2 t} \left\{ \begin{bmatrix} q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \right.$$

2 'modes' or patterns

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1}$$

GETTING P

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\therefore AP = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\therefore A\left[\begin{array}{c:c} p_{11} & p_{12} \\ p_{21} & p_{22} \end{array}\right] = \left[\begin{array}{c:c} p_{11} & p_{12} \\ p_{21} & p_{22} \end{array}\right]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Q1. HOW MUCH OF EACH MODE IS EXCITED?

Q2. IF A MODE IS EXCITED, HOW MUCH OF IT IS SEEN IN x_1 & x_2?

MODE $e^{\lambda_1 t} \longrightarrow \left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} \begin{matrix} P_{11} \\ P_{21} \end{matrix}$

$e^{\lambda_2 t} \longrightarrow \left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} \begin{matrix} P_{12} \\ P_{22} \end{matrix}$

$$A \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \lambda_1 \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \leftarrow p_1 \quad \checkmark$$

$$\& \quad A \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \lambda_2 \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \quad \checkmark$$

Eigenvalues & right eigenvectors.

$$\dot{y} = P^{-1}AP\,y = \Lambda\,y$$

$$P^{-1}AP = \Lambda = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

$$y = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}$$

$$y(0) = P^{-1}x(0)$$

$$x = Py = \begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}^{-1}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

has to exist

$\therefore (A - \lambda I)\, p = \underline{0}$

column of P.

$p_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$

Trivial $\rightarrow$ $p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\rightarrow$ NOT ACCEPTABLE

if $(A - \lambda I)$ is nonsingular $\}\, \times$

$\therefore \det(A - \lambda I) = 0$ characteristic equation.

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = 0 \quad \longleftarrow \quad \underline{\underline{\det(A - \lambda I)}}.$$

$$\therefore \underline{(a_{11} - \lambda)(a_{22} - \lambda) - a_{21} a_{22} = 0}$$

$$\underline{2 \text{ solutions. } : \lambda_1 , \lambda_2 .}$$

LARGER SYSTEMS ?

↓

NUMERICAL (iterative). 2X2 [A] 20X20

$$(A - \lambda_1 I)\, p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$p_1 \rightarrow$ NOT A UNIQUE SOLUTION

αp_1 is also a solution

$$P = \left[\, p_1 \;\vdots\; p_2 \,\right]$$

EXAMPLE

$$P = \left[\, \alpha p_1 \;\vdots\; p_2 \,\right]$$

$$= \left[\, p_1 \;\vdots\; \alpha_2 p_2 \,\right]$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}}_{\text{'A'}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix} = \det(A - \lambda I)$$

$$= 0.$$

$$(1-\lambda)^2 - 0.5 \times 0.5 = 0.$$

$$\lambda^2 - 2\lambda + 1 - 0.25 = 0$$

$$\lambda^2 - 2\lambda + 0.75 = 0.$$

$$\lambda^2 - 2\lambda + 0.75 = 0$$

$$\Rightarrow (\lambda - 0.5)(\lambda - 1.5) = 0$$

$$\lambda_1 = 0.5$$

$$\lambda_2 = \cancel{0}\ 1.5$$

p_1

$$\begin{bmatrix} 1-0.5 & 0.5 \\ 0.5 & 1-0.5 \end{bmatrix} p_1 = \underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(A - \lambda_1 I)$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

'p'

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $p_{11} = 1$

$$0.5 \times 1 + 0.5\, p_{21} = 0$$

$$\Rightarrow p_{21} = -1$$

$$p_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow \lambda_1 = 0.5$$

$$\cancel{\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & \end{bmatrix}} \cancel{\begin{bmatrix} 1-1.5 & 0.5 \\ 0.5 & 1-1.5 \end{bmatrix}} \cancel{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}.$$

$$\underbrace{\begin{bmatrix} 1-1.5 & 0.5 \\ 0.5 & 1-1.5 \end{bmatrix}}_{\uparrow\, A - \lambda_2 I} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P_{12} = 1$$

$$\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ P_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad P_{22} = +1$$

$$\begin{bmatrix} +1 \\ +1 \end{bmatrix} \qquad \rightarrow \quad \lambda_2 = 1.5$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{p_1} e^{+0.5t} k_1 + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{p_2} e^{1.5t} k_2$$

$$k_1 = \begin{bmatrix} q_{11} & q_{12} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

$$K_2 = \begin{bmatrix} q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad PP^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

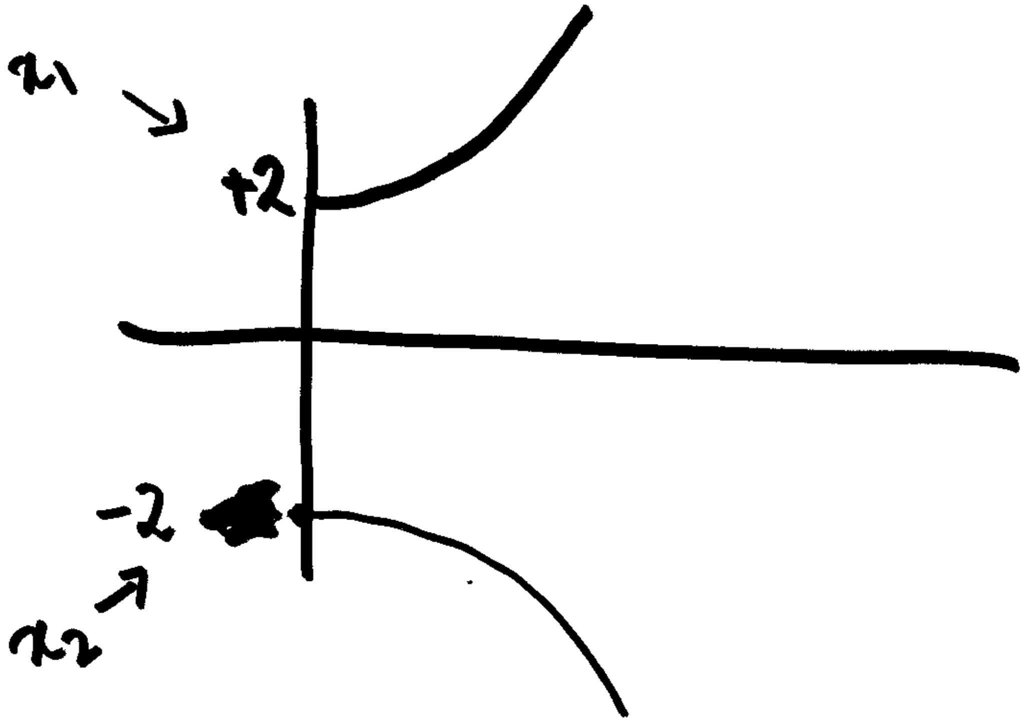
$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix} \checkmark = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{0.5t} \underbrace{\frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{`2'}}$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1.5t} \underbrace{\frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{0}$$

$x_1(0) = +2$ $x_1(0) = 2$

$x_2(0) = -2.$ $x_2(0) = 2$


x1
+2
-2
x2

ISSUES

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{eigenvalues?} \qquad \dot{x} = A x$$

$$\underline{\det(A - \lambda I) = (1-\lambda)^2 = 0}$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 1$$

p_1 & p_2 $\quad P$

$$\dot{x}_1(t) = x_1(t) + x_2(t)$$

$$\dot{x}_2(t) = x_2(t)$$

$$x_2(t) = e^{t} x_2(0)$$

$$\dot{x}_1(t) = \underbrace{x_1(t)}_{ax} + \underbrace{e^{t} x_2(0)}_{bu.}$$

$$\dot{x}_1(t) = x_1(t) + e^{t} x_2(0)$$

$$x_1(t) = e^{t} x_1(0) + \int_0^t e^{(t-\tau)} e^{\tau} x_2(0)\, d\tau.$$

$$= e^{t} x_1(0) + e^{t} \int_0^t x_2(0)\, d\tau.$$

$$= e^{t} x_1(0) + \underline{\underline{t e^{t} x_2(0)}}.$$

STABILITY

$$\mathrm{Re}(\lambda) < 0 \implies \text{STABLE}$$

(RESPONSE)* → if A is diagonalizable

$$x(t) = \sum_{i=1}^{n} p_i \, e^{\lambda_i t} \, q_i^T x(0) = e^{At} x(0).$$

q_i^T → row of the inverse of

$$P = [\, p_1 \,\vdots\, p_2 \,\vdots\, \cdots \,\vdots\, p_n \,]$$