Numerical Optimization

Linear Programming - Duality

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NPTEL Course on Numerical Optimization

The Diet Problem: Find the *most economical* diet that satisfies *minimum* nutritional requirements.

- Number of food items: n
- Number of nutritional ingredient: m
- Each person must consume *at least* b_j units of nutrient j per day
- Unit cost of food item i: c_i
- Each unit of food item i contains a_{ji} units of the nutrient j
- Number of units of food item i consumed: x_i

Constraint corresponding to the nutrient j:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \geq b_j, \quad x_i \geq 0 \; \forall \; i$$

Cost:

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

The Diet Problem:

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \geq b_j \; \forall \; j \\ & x_i \geq 0 \; \forall \; i \end{aligned}$$

Given: $\boldsymbol{c} = (c_1, \ldots, c_n)^T, \boldsymbol{A} = (\boldsymbol{a}_1 | \ldots | \boldsymbol{a}_n), \boldsymbol{b} = (b_1, \ldots, b_m)^T$.
Consider the following situation:

- Unit cost of each vitamin pill: λ_j, $\lambda_j \geq 0 \; \forall \; j$
- Each person must consume *at least* b_j units of nutrient j per day
- Cost: $\lambda_1 b_1 + \ldots + \lambda_m b_m$
- Ensure that the price for a nutrient mixture substitute for food item i should be at the most c_i

$$\sum_{j=1}^{m} a_{ij}\lambda_j \leq c_i \; \forall \; i$$

The problem,

$$\begin{aligned} \max \quad & \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n \\ \text{s.t.} \quad & a_{i1}\lambda_1 + a_{i2}\lambda_2 + \ldots + a_{im}\lambda_m \leq c_i \ \forall\ i \\ & \lambda_j \geq 0 \ \forall\ j \end{aligned}$$

is the *dual problem* of

$$\begin{aligned} \min \quad & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\ \text{s.t.} \quad & a_{j1} x_1 + a_{j2} x_2 + \ldots + a_{jn} x_n \geq b_j \ \forall\ j \\ & x_i \geq 0 \ \forall\ i \end{aligned}$$

Duality in Linear Programming

- Symmetric Form of Duality

Primal Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \boldsymbol{b}^T \boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{\lambda} \leq \boldsymbol{c} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0} \end{aligned}$$

- Asymmetric form of Duality

Primal Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \boldsymbol{b}^T \boldsymbol{\mu} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{\mu} \leq \boldsymbol{c} \end{aligned}$$

Primal Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \boldsymbol{b}^T\boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^T\boldsymbol{\lambda} \leq \boldsymbol{c} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0} \end{aligned}$$

For linear programs, the dual of the dual is the primal problem.

Primal Problem

$$\begin{aligned} -\min \quad & -\boldsymbol{b}^T\boldsymbol{\lambda} \\ \text{s.t.} \quad & -\boldsymbol{A}^T\boldsymbol{\lambda} \geq -\boldsymbol{c} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Consider the following primal and dual problems:

Primal Problem (P)

$$\begin{array}{rl} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Dual Problem (D)

$$\begin{array}{rl} \max & \boldsymbol{b}^T\boldsymbol{\mu} \\ \text{s.t.} & A^T\boldsymbol{\mu} \leq \boldsymbol{c} \end{array}$$

Weak Duality Theorem

If $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are primal and dual feasible respectively, then $\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{\mu}$.

Strong Duality Theorem

If either of the problems **P** or **D** has a finite optimal solution, so does the other, and the corresponding optimal objective function values are equal. If any of these two problems is unbounded, the other problem has no feasible solution.

	Minimization Problem		Maximization Problem	
variables	<= 0	⟷	>=	constraints
	>= 0	⟷	<=	
	unrestricted	⟷	=	
constraints	<=	⟷	<= 0	variables
	>=	⟷	>= 0	
	=	⟷	unrestricted	

Relationships between primal and dual problems

Example:

Primal Problem

$$
\begin{array}{rc}
\min & 3x_1 - 5x_2 + x_3 \\
\text{s.t.} & x_1 - 2x_3 \geq 4 \\
 & 2x_1 - x_2 + x_3 \geq 2 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

Dual Problem

$$
\begin{array}{rc}
\max & 4y_1 + 2y_2 \\
\text{s.t.} & y_1 + 2y_2 \leq 3 \\
 & -y_2 \leq -5 \\
 & -2y_1 + y_2 \leq 1 \\
 & y_1, y_2 \geq 0
\end{array}
$$

Primal problem is unbounded and the dual problem is infeasible

Example:

Primal Problem

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} \min \quad & y_1 - 2y_2 \\ \text{s.t.} \quad & y_1 - y_2 \geq 1 \\ & -y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Both primal and dual problems are infeasible

Example:

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & -2x_1 + 5x_2 - x_3 + 3x_4 \leq -3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The dual problem is

$$\begin{aligned} \max \quad & 2y_1 - 3y_2 \\ \text{s.t.} \quad & y_1 - 2y_2 \leq 2 \\ & 6y_1 + 5y_2 \leq 15 \\ & 3y_1 - y_2 \leq 5 \\ & y_1 + 3y_2 \leq 6 \\ & y_1 \geq 0, \; y_2 \leq 0 \end{aligned}$$

Solution of the primal problem using Simplex Method:

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & -2x_1 + 5x_2 - x_3 + 3x_4 \leq -3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The equivalent problem is:

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & 2x_1 - 5x_2 + x_3 - 3x_4 \geq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Phase I: Introducing artificial variables, the constraints become

$$\begin{aligned} x_1 + 6x_2 + 3x_3 + x_4 - x_5 + x_6 &= 2 \\ 2x_1 - 5x_2 + x_3 - 3x_4 - x_7 + x_8 &= 3 \\ x_j \geq 0,\ j &= 1, \ldots, 8 \end{aligned}$$

Therefore, the artificial linear program is,

$$\begin{aligned} \min \quad & x_6 + x_8 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 - x_5 + x_6 = 2 \\ & 2x_1 - 5x_2 + x_3 - 3x_4 - x_7 + x_8 = 3 \\ & x_j \geq 0,\ j = 1, \ldots, 8 \end{aligned}$$

Initial Tableau:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	6	3	1	-1	1	0	0	2
2	-5	1	-3	0	0	-1	1	3
0	0	0	0	0	1	0	1	0

Making the relative costs of basic variables 0,

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	6	3	1	-1	1	0	0	2
2	-5	1	-3	0	0	-1	1	3
-3	-1	-4	2	1	0	1	0	-5

Using Simplex Method, final tableau for the artificial linear program:

$$\left(\begin{array}{cccccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \text{RHS} \\
\hline\hline
0 & \frac{17}{5} & 1 & 1 & -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\
1 & -\frac{21}{5} & 0 & -2 & \frac{1}{5} & -\frac{1}{5} & -\frac{3}{5} & 0 & \frac{7}{5} \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right)$$

Basic variables for the original program: $x_1 = \frac{7}{5}, x_3 = \frac{1}{5}$
Initial Tableau (for the original program):

$$\left(\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & \text{RHS} \\
\hline\hline
0 & \frac{17}{5} & 1 & 1 & -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\
1 & -\frac{21}{5} & 0 & -2 & \frac{1}{5} & -\frac{3}{5} & \frac{7}{5} \\
\hline
2 & 15 & 5 & 6 & 0 & 0 & 0
\end{array}\right)$$

Making the relatives costs of basic variables 0,

$$\left(\begin{array}{cccccc|c} \color{red}{x_1} & x_2 & \color{red}{x_3} & x_4 & x_5 & x_7 & \text{RHS} \\ \hline \hline 0 & \frac{17}{5} & 1 & 1 & -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & -\frac{21}{5} & 0 & -2 & \frac{1}{5} & -\frac{3}{5} & \frac{7}{5} \\ \hline 0 & \frac{32}{5} & 0 & 5 & \frac{8}{5} & \frac{1}{5} & -\color{blue}{\frac{19}{5}} \end{array}\right)$$

Primal Problem

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & -2x_1 + 5x_2 - x_3 + 3x_4 \leq -3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & 2y_1 - 3y_2 \\ \text{s.t.} \quad & y_1 - 2y_2 \leq 2 \\ & 6y_1 + 5y_2 \leq 15 \\ & 3y_1 - y_2 \leq 5 \\ & y_1 + 3y_2 \leq 6 \\ & y_1 \geq 0,\ y_2 \leq 0 \end{aligned}$$

Optimal objective function = $\frac{19}{5}$ (for both the problems)

Consider the following primal and dual problems:

Primal Problem (P)

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

Dual Problem (D)

$$\begin{aligned} \max \quad & \boldsymbol{b}^T \boldsymbol{\mu} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{\mu} \leq \boldsymbol{c} \end{aligned}$$

Theorem

Let **P** *have an optimal basic feasible solution,* $(\boldsymbol{B}^{-1}\boldsymbol{b}, \mathbf{0})$ *corresponding to the basis* B*. Then,* $\boldsymbol{\mu}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ *is an optimal solution to the dual problem* **D** *and the optimal values of both problems are equal.*

Proof.

$\boldsymbol{x} = (\boldsymbol{B}^{-1}\boldsymbol{b}, \boldsymbol{0})$ is an optimal basic feasible solution. At optimality, KKT conditions are satisfied. Therefore,

$$\boldsymbol{\lambda}_B^T = \boldsymbol{0}^T, \;\; \boldsymbol{\lambda}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} \geq \boldsymbol{0}^T \;\Rightarrow\; \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} \leq \boldsymbol{c}_N^T$$

Define, $\boldsymbol{\mu}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$.
$\therefore \boldsymbol{\mu}^T\boldsymbol{A} = \boldsymbol{\mu}^T(\boldsymbol{B}, \;\; \boldsymbol{N}) = (\boldsymbol{c}_B^T, \; \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N}) \leq (\boldsymbol{c}_B^T, \; \boldsymbol{c}_N^T) = \boldsymbol{c}^T$
Therefore, $\boldsymbol{\mu}$ is dual feasible.
By *Weak Duality Theorem*, $\boldsymbol{\mu}^T\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{c}^T\boldsymbol{x} \Rightarrow \boldsymbol{\mu}^T\boldsymbol{b} \leq \boldsymbol{c}^T\boldsymbol{x}$.
Further, $\boldsymbol{\mu}^T\boldsymbol{b} = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B^T\boldsymbol{x}_B = \boldsymbol{c}^T\boldsymbol{x}$.
Thus, optimal values of **P** and **D** are equal. □

How to obtain optimal $\boldsymbol{\mu}$ after solving the primal problem?

LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

$$\left(\begin{array}{c|c|c|c} \text{Basic Variables} & \text{Nonbasic Variables} & \text{Artificial Variables} & \text{RHS} \\ \hline \hline \boldsymbol{B} & \boldsymbol{N} & \boldsymbol{I} & \boldsymbol{b} \\ \hline \boldsymbol{c}_B^T & \boldsymbol{c}_N^T & \boldsymbol{0}^T & 0 \end{array} \right)$$

$$\left(\begin{array}{c|c|c|c} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{B}^{-1} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \hline \boldsymbol{c}_B^T & \boldsymbol{c}_N^T & \boldsymbol{0}^T & 0 \end{array} \right)$$

$$\left(\begin{array}{c|c|c|c} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{B}^{-1} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \hline \boldsymbol{0}^T & \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{array} \right)$$

LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

$$\left(\begin{array}{c|c|c|c} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{B}^{-1} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \hline \boldsymbol{0}^T & \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{array}\right)$$

At optimality of primal problem:

- $\boldsymbol{\lambda}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} \geq \boldsymbol{0}^T$
- $\boldsymbol{\mu}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$ is an optimal solution to the dual problem

Consider the problem,

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual problem:

$$\begin{aligned} \max \quad & 2\lambda_1 + \lambda_2 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 \leq -3 \\ & \lambda_1 \leq -1 \\ & \lambda_1, \lambda_2 \leq 0 \end{aligned}$$

- Optimal primal objective function = -4 at $\boldsymbol{x}^* = (1, 1)^T$
- Optimal dual objective function = -4 at $\boldsymbol{\lambda}^* = (-1, -2)^T$

$$
\begin{aligned}
\min \quad & -3x_1 - x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 2 \\
& x_1 + x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

- Initial Basic Feasible Solution:
  $\boldsymbol{x}_B = (x_3, x_4)^T = (2, 1)^T$, $\boldsymbol{x}_N = (x_1, x_2)^T = (0, 0)^T$

Initial Tableau:

x_1	x_2	x_3	x_4	RHS
1	1	1	0	2
1	0	0	1	1
-3	-1	0	0	0

Initial Tableau:

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

Final Tableau:

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

- Optimal primal solution: $\boldsymbol{x}^* = (1, 1)^T$
- Optimal dual solution: $\boldsymbol{\lambda}^* = (-1, -2)^T$