Numerical Optimization
Linear Programming

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NPTEL Course on Numerical Optimization
Transportation Problem

\[
\begin{align*}
\min_{x} & \quad \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j=1}^{3} x_{ij} \leq a_i, \quad i = 1, 2 \\
& \quad \sum_{i=1}^{2} x_{ij} \geq b_j, \quad j = 1, 2, 3 \\
& \quad x_{ij} \geq 0 \quad \forall \ i, j
\end{align*}
\]

- \( a_i \): Capacity of the plant \( F_i \)
- \( b_j \): Demand of the outlet \( R_j \)
- \( c_{ij} \): Cost of shipping one unit of product from \( F_i \) to \( R_j \)
- \( x_{ij} \): Number of units of the product shipped from \( F_i \) to \( R_j \) (variables)

The objective is to minimize
\[
\sum_{ij} c_{ij} x_{ij}
\]
\[
\sum_{j=1}^{3} x_{ij} \leq a_i, \quad i = 1, 2 \quad \text{(constraints)}
\]
\[
\sum_{i=1}^{2} x_{ij} \geq b_j, \quad j = 1, 2, 3 \quad \text{(constraints)}
\]
\[
x_{ij} \geq 0 \quad \forall \ i, j \quad \text{(constraints)}
\]
The Diet Problem: Find the most economical diet that satisfies minimum nutritional requirements.

- Number of food items: \( n \)
- Number of nutritional ingredient: \( m \)
- Each person must consume at least \( b_j \) units of nutrient \( j \) per day
- Unit cost of food item \( i \): \( c_i \)
- Each unit of food item \( i \) contains \( a_{ji} \) units of the nutrient \( j \)
- Number of units of food item \( i \) consumed: \( x_i \)

Constraint corresponding to the nutrient \( j \):

\[
a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \geq b_j, \quad x_i \geq 0 \forall i
\]

Cost:

\[
c_1x_1 + c_2x_2 + \ldots + c_nx_n
\]
Problem:

\[
\begin{align*}
\min \quad & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{s.t.} \quad & a_{j1} x_1 + a_{j2} x_2 + \ldots + a_{jn} x_n \geq b_j \ \forall \ j \\
& x_i \geq 0 \ \forall \ i
\end{align*}
\]

Given: \( c = (c_1, \ldots, c_n)^T, A = (a_1 | \ldots | a_n), b = (b_1, \ldots, b_m)^T \).

Linear Programming Problem (LP):

\[
\begin{align*}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \geq b \\
& x \geq 0
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \).

- Assumption: \( m \leq n, \text{rank}(A) = m \)
- Linear Constraints can be of the form \( Ax = b \) or \( Ax \leq b \)
Constraint (Feasible) Set:

- Inequality constraint of the type \( \{ x : a^T x \leq b \} \) or \( \{ x : a^T x \geq b \} \) denotes a *half space*
- Equality constraint, \( \{ x : a^T x = b \} \), represents an affine space
- Non-negativity constraint, \( x \geq 0 \)
- Constraint set of an LP is a *convex* set

Polyhedral Set

\[
X = \{ x : Ax \leq b, \ x \geq 0 \}
\]

Polytope: A bounded polyhedral set
Consider the constraint set in $\mathbb{R}^2$:

\[
\{ (x_1, x_2) : x_1 + x_2 \leq 2, x_1 \leq 1, x_1 \geq 0, x_2 \geq 0 \}
\]
Consider the constraint set in $\mathbb{R}^2$:

$$\{ (x_1, x_2) : x_1 \geq 1, x_2 \geq x_1 \}$$
Feasible set can be a singleton set

Feasible Set = \{(x_1, x_2) : x_1 + x_2 = 2, -x_1 + x_2 = 1\} = \{A\}
Feasible set can be empty!

Feasible Set = \{ (x_1, x_2) : x_1 + x_2 \geq 2, x_1 + x_2 \leq 1 \} = \emptyset
Definition

Let $X$ be a convex set. A point $x \in X$ is said to be an **extreme point** (corner point or vertex) of $X$ if $x$ cannot be represented as a strict convex combination of two distinct points in $X$.

Extreme Points: A, B, C and D.

*E is not an extreme point.*
Extreme Point: A
- Constraint Set:
  \[ X = \{ (x_1, x_2) : x_1 + x_2 \leq 2, x_1 \leq 1, x_1 \geq 0, x_2 \geq 0 \} \]
- 4 constraints in \( \mathbb{R}^2 \)

- Two constraints are *binding* (active) at every extreme point
- Fewer than two constraints are binding at other points
Consider the constraint set: \( X = \{ x : Ax \leq b, x \geq 0 \} \) where \( A \in \mathbb{R}^{m \times n} \) and \( \text{rank}(A) = m \).

- \( m + n \) hyperplanes associated with \( m + n \) halfspaces
- \( m + n \) halfspaces define \( X \)
- An extreme point lies on \( n \) linearly independent defining hyperplanes of \( X \)
- If \( X \) is nonempty, the set of extreme points of \( X \) is not empty and has a finite number of points.
- An edge of \( X \) is formed by intersection of \( n - 1 \) linearly independent hyperplanes
- Two extreme points of \( X \) are said to be adjacent if the line segment joining them is an edge of \( X \)
For example, B and C are adjacent points.

Adjacent extreme points have $n - 1$ common binding linearly independent hyperplanes.
Remarks:
Consider the constraint set: $X = \{x : Ax = b, x \geq 0\}$ where $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = m$.

- Let $\bar{x} \in X$ be an extreme point of $X$
- $m$ equality constraints are active at $\bar{x}$
- Therefore, $n - m$ additional hyperplanes (from the non-negativity constraints) are active at $\bar{x}$
Geometric Solution of a LP:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \).
Objective Function Decreases
$c^T x = \beta$  $\quad (\beta < \alpha)$

$c^T x = \alpha$

$c^T x = 0$
$\mathbf{c}^T \mathbf{x} = 0$
Set of Optimal Solutions

$c^T x = \beta \quad (\beta < \alpha)$

$c^T x = 0$
Example:

\[
\begin{align*}
\text{min} & \quad -2x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 5 \\
& \quad x_1 + 2x_2 \leq 6
\end{align*}
\]
Example:

\[
\begin{align*}
\text{min} & \quad -2x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 5 \\
& \quad x_1 + 2x_2 \leq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Example:

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\text{min} & \quad -2x_1 - x_2 \\
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Example:

$$\begin{align*}
\text{min} & \quad -2x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 5 \\
& \quad x_1 + 2x_2 \leq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}$$
Example:

\[ \min \quad c^T x \]

\[ \text{s.t.} \quad x \in X \]
Example:

$$\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad x \in X
\end{align*}$$

Unbounded Feasible Set, $X$

Optimal Solution
Example:

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad x \in X
\end{align*}
\]
Example:

\[ \min \ c^T x \]
\[ \text{s.t. } x \in X \]
Consider a linear programming problem \textbf{LP}:

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad a_i^T x \ (\leq, =, \geq) \ b_i, \ i = 1, \ldots, m \\
& \quad x \geq 0
\end{align*}
\]

Let \( X = \{x : a_i^T x \ (\leq, =, \geq) \ b_i, \ i = 1, \ldots, m, x \geq 0\} \).

Remarks:

- \( X \) is a closed convex set
- The set of optimal solutions is a convex set.
- The linear program may have \textit{no solution} or a \textit{unique solution} or \textit{infinitely many solutions}.
- If \( x^* \) is an optimal solution to \textbf{LP}, then \( x^* \) must be a \textit{boundary point} of \( X \). If \( z = c^T x^* \), then \( \{x : c^T x = z\} \) is a supporting hyperplane to \( X \).
- If \( X \) is compact and if there is an optimal solution to \textbf{LP}, then \textit{at least one} extreme point of \( X \) is an optimal solution to the linear programming problem.
LP in Standard Form:

\[
\min \quad c^T x \\
\text{s.t.} \quad Ax = b \\
\quad x \geq 0
\]

where \( A \in \mathbb{R}^{m \times n} \) and \( \text{rank}(A) = m \).

Assumption: Feasible set is non-empty
Any linear program can be converted to the Standard Form.

(a) \[ \text{max } c^T x = - \text{min } -c^T x \]

(b) Constraint of the type

\[ a^T x \leq b, \quad x \geq 0 \]

can be written as

\[ a^T x + y = b \\
\quad x \geq 0 \\
\quad y \geq 0 \]
(c) Constraint of the type

\[ a^T x \geq b, \ x \geq 0 \]

can be written as

\[ a^T x - z = b \]

\[ x \geq 0 \]

\[ z \geq 0 \]

(d) Free variables \((x_i \in \mathbb{R})\) can be defined as

\[ x_i = x_i^+ - x_i^-, \ x_i^+ \geq 0, \ x_i^- \geq 0 \]
Example:

\[
\begin{align*}
\text{min} & \quad x_1 - 2x_2 - 3x_3 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 \leq 14 \\
& \quad x_1 + 2x_2 + 4x_3 \geq 12 \\
& \quad x_1 - x_2 + x_3 = 2 \\
& \quad x_1, x_2 \text{ unrestricted}
\end{align*}
\]

- Write the constraints as equality constraints
  - \( x_1 + 2x_2 + x_3 + x_4 = 14, \ x_4 \geq 0 \)
  - \( x_1 + 2x_2 + 4x_3 - x_5 = 12, \ x_5 \geq 0 \)

- Define new variables \( x_1^+, x_1^-, x_2^+, x_2^- \) and \( x_3' \) such that
  - \( x_1 = x_1^+ - x_1^- \), where \( x_1^+ \geq 0, x_1^- \geq 0 \)
  - \( x_2 = x_2^+ - x_2^- \), where \( x_2^+ \geq 0, x_2^- \geq 0 \)
  - \( x'_3 = -3 - x_3 \) so that \( x'_3 \geq 0 \)
Therefore, the program

\[
\begin{align*}
\text{min} & \quad x_1 - 2x_2 - 3x_3 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 \leq 14 \\
& \quad x_1 + 2x_2 + 4x_3 \geq 12 \\
& \quad x_1 - x_2 + x_3 = 2 \\
& \quad x_1, x_2 \text{ unrestricted} \\
& \quad x_3 \leq -3
\end{align*}
\]

can be converted to the standard form:

\[
\begin{align*}
\text{min} & \quad x_1^+ - x_1^- - 2(x_2^+ - x_2^-) + 3(3 + x_3') \\
\text{s.t.} & \quad x_1^+ - x_1^- + 2(x_2^+ - x_2^-) - x_3' + x_4 = 17 \\
& \quad x_1^+ - x_1^- + 2(x_2^+ - x_2^-) - 4x_3' - x_5 = 24 \\
& \quad x_1^+ - x_1^- - x_2^+ + x_2^- - x_3' = 5 \\
& \quad x_1^+, x_1^-, x_2^+, x_2^-, x_3', x_4, x_5 \geq 0
\end{align*}
\]
Consider the linear program in standard form (SLP):

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n} \), \( \text{rank}(A) = \text{rank}(A|b) = m \).
Let \( B \in \mathbb{R}^{m \times m} \) be formed using \( m \) linearly independent columns of \( A \).
Therefore, the system of equations, \( Ax = b \) can be written as,

\[
(B \quad N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b.
\]

Letting \( x_N = 0 \), we get

\[
Bx_B = b \quad \Rightarrow \quad x_B = B^{-1}b. \quad (x_B : \text{Basic Variables})
\]

\((x_B \quad 0)^T: \text{Basic solution w.r.t. the basis matrix } B\)
Basic Feasible Solution

If \( x_B \geq 0 \), then \((x_B \ 0)^T\) is called a *basic feasible solution* of

\[
Ax = b \\
x \geq 0
\]

w.r.t. the basis matrix \( B \).

**Theorem**

Let \( X = \{x : Ax = b, \ x \geq 0\} \). \( x \) is an extreme point of \( X \) if and only if \( x \) is a basic feasible solution of

\[
Ax = b \\
x \geq 0.
\]
Proof.

(a) Let \( x \) be a basic feasible solution of \( Ax = b, x \geq 0 \).

Therefore, \( x = (x_1, \ldots, x_m, 0, \ldots, 0) \). Let \( B = (a_1|a_2|\ldots|a_m) \)

\[ \begin{array}{c}
\geq 0 \\
n-m
\end{array} \]

where \( a_1, \ldots, a_m \) are linearly independent. So,

\[ x_1a_1 + \ldots + x ma_m = b. \]

Suppose \( x \) is not an extreme point of \( X \).
Let \( y, z \in X, y \neq z \) and \( x = \alpha y + (1 - \alpha)z, 0 < \alpha < 1 \).
Since \( y, z \geq 0 \), we have

\[ \begin{array}{c}
y_{m+1} = \ldots = y_n = 0 \\
z_{m+1} = \ldots = z_n = 0
\end{array} \]

and

\[ y_1a_1 + \ldots + y ma_m = b \]

\[ z_1a_1 + \ldots + z ma_m = b \]

Since \( a_1, \ldots, a_m \) are linearly independent, \( x = y = z \), a contradiction. So, \( x \) is an extreme point of \( X \).
Proof.(continued)

(b) Let \( x \) be an extreme point of \( X \).

\[ x \in X \implies Ax = b, \ x \geq 0. \]

There exist \( n \) linearly independent constraints active at \( x \).

- \( m \) active constraints associated with \( Ax = b \).
- \( n - m \) active constraints associated with \( n - m \) non-negativity constraints

\( x \) is the unique solution of \( Ax = b, \ x_N = 0. \)

\[
Ax = b \implies Bx_B + Nx_N = b \implies x_B = B^{-1}b \geq 0
\]

Therefore, \( x = (x_B \ x_N)^T \) is a basic feasible solution. \( \square \)

Number of basic solutions \( \leq \binom{n}{m} \)

Enough to search the finite set of vertices of \( X \) to get an optimal
Theorem

Let $X$ be non-empty and compact constraint set of a linear program. Then, an optimal solution to the linear program exists and it is attained at a vertex of $X$.

Proof.

Objective function, $c^T x$, of the linear program is continuous and the constraint set is compact. Therefore, by Weierstrass’ Theorem, optimal solution exists. The set of vertices, $\{x_1, \ldots, x_k\}$, of $X$ is finite. Therefore, $X$ is the convex hull of $x_1, \ldots, x_k$. Hence, for every $x \in X$, $x = \sum_{i=1}^{k} \alpha_i x_i$ where $\alpha_i \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$. Let $z^* = \min_{1 < i \leq k} c^T x_i$. Therefore, for any $x \in X$, $z = c^T x = \sum_{i=1}^{k} \alpha_i c^T x_i \geq z^* \sum_{i=1}^{k} \alpha_i = z^*$. So, the minimum value of $c^T x$ is attained at a vertex of $X$. 
Consider the constraints:

\[ x_1 + x_2 \leq 2 \]
\[ x_1 \leq 1 \]
\[ x_1, x_2 > 0 \]
The given constraints

\[ x_1 + x_2 \leq 2 \]
\[ x_1 \leq 1 \]
\[ x_1, x_2 \geq 0 \]

can be written in the form, \( Ax = b, \ x \geq 0 \):

\[ x_1 + x_2 + x_3 = 2 \]
\[ x_1 + x_4 = 1 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

Let \( A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1 | a_2 | a_3 | a_4) \) and \( b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1|a_2|a_3|a_4) \] and \[ b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

(1) \[ B = (a_1|a_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ x_B = (x_1 \ x_2)^T = B^{-1}b = (1 \ 1)^T \] and \[ x_N = (x_3 \ x_4)^T = (0 \ 0)^T. \]

\[ x = (x_B \ x_N)^T \] corresponds to the vertex C.
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1 | a_2 | a_3 | a_4) \text{ and } b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} . \]

(2) \[ B = (a_1 | a_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ x_B = (x_1 \ x_3)^T = B^{-1} b = (1 \ 1)^T \text{ and } x_N = (x_2 \ x_4)^T = (0 \ 0)^T. \]

\[ x = (x_B \ x_N)^T \] corresponds to the vertex \( B \).
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1 | a_2 | a_3 | a_4) \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

\[(3) \quad B = (a_1 | a_4) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ x_B = (x_1 \ x_4)^T = B^{-1}b = (2 \ -1)^T \quad \text{and} \quad x_N = (x_2 \ x_3)^T = (0 \ 0)^T. \]

\[ x = (x_B \ x_N)^T \] is not a basic feasible point
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1 | a_2 | a_3 | a_4) \text{ and } b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

(4) \[ B = (a_2 | a_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ x_B = (x_2 \ x_4)^T = B^{-1}b = (2 \ 1)^T \text{ and } x_N = (x_1 \ x_3)^T = (0 \ 0)^T. \]

\[ x = (x_B \ x_N)^T \] corresponds to the vertex \( D \).
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (a_1|a_2|a_3|a_4) \text{ and } b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

\begin{align*}
(5) \quad B &= (a_3|a_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 x_B &= (x_3 \ x_4)^T = B^{-1}b = (2 \ 1)^T \text{ and } x_N = (x_1 \ x_2)^T = (0 \ 0)^T. \\
 x &= (x_B \ x_N)^T \text{ corresponds to the vertex } A.
\end{align*}
Example:

\[ \begin{align*} 
\text{min} & \quad -3x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 2 \\
& \quad x_1 \leq 1 
\end{align*} \]