Numerical Optimization

Convex Sets

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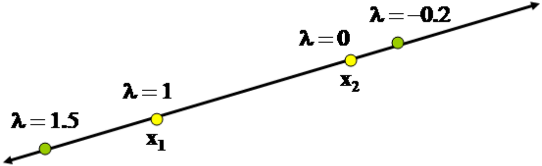
NPTEL Course on Numerical Optimization

Line and line segment

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n, \mathbf{x}_1 \neq \mathbf{x}_2$.



Line passing through $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \quad \lambda \in \mathbb{R}\}$$

Line Segment, $LS[\mathbf{x}_1, \mathbf{x}_2]$:

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \quad \lambda \in [0, 1]\}$$

Affine sets

Definition

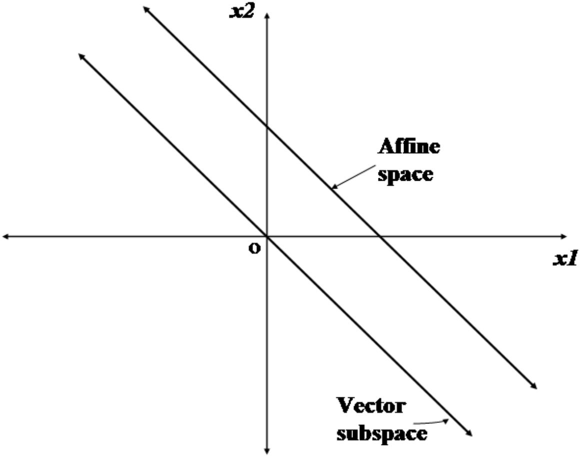
A set $X \subseteq \mathbb{R}^n$ is affine if for any $\mathbf{x}_1, \mathbf{x}_2 \in X$, and $\lambda \in \mathbb{R}$,

$$\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in X.$$

If $X \subseteq \mathbb{R}^n$ is an affine set, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in X$ and $\sum_i \lambda_i = 1$, then the point $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_k \mathbf{x}_k \in X$.

Examples

- Solution set of linear equations: $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$.
- A subspace or a translated subspace


x2
Affine
space
o
x1
Vector
subspace

Result

If X is an affine set and $\mathbf{x}_0 \in X$, then $\{\mathbf{x} - \mathbf{x}_0 : \mathbf{x} \in X\}$ forms a subspace.

Proof.

Let $Y = \{\mathbf{x} - \mathbf{x}_0 : \mathbf{x} \in X\}$. To show that Y is a subspace, we need to show, $\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 \in Y$ for any $\mathbf{y}_1, \mathbf{y}_2 \in Y$ and $\alpha, \beta \in \mathbb{R}$.
Let $\mathbf{y}_1, \mathbf{y}_2 \in Y$ and $\alpha, \beta \in \mathbb{R}$.
Therefore, $\mathbf{y}_1 + \mathbf{x}_0 \in X$ and $\mathbf{y}_2 + \mathbf{x}_0 \in X$.
$\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 = \alpha(\mathbf{y}_1 + \mathbf{x}_0) + \beta(\mathbf{y}_2 + \mathbf{x}_0) - (\alpha + \beta)\mathbf{x}_0$.
$\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 + \mathbf{x}_0 = \alpha(\mathbf{y}_1 + \mathbf{x}_0) + \beta(\mathbf{y}_2 + \mathbf{x}_0) + (1 - \alpha - \beta)\mathbf{x}_0$.
Since X is an affine set, $\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 + \mathbf{x}_0 \in X$.
Thus, $\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 \in Y \Rightarrow Y$ is a subspace.

$\square$

Definition

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. A point $\mathbf{x}$ is said to be an affine combination of points in X if

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Definition

Let $X \subseteq \mathbb{R}^n$. The set of all affine combinations of points in X is called the affine hull of X.

$$\text{aff}(X) = \{\sum_{i=1}^{k} \lambda_i \mathbf{x}_i : \mathbf{x}_1, \ldots, \mathbf{x}_k \in X, \sum_{i} \lambda_i = 1\}$$

Convex Sets

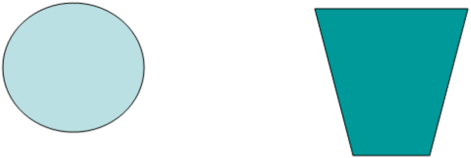
Definition

A set $C \subseteq \mathbb{R}^n$ is convex if for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and any scalar λ with $0 \leq \lambda \leq 1$, we have

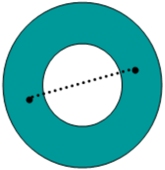
$$\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in C.$$

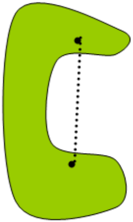
Examples

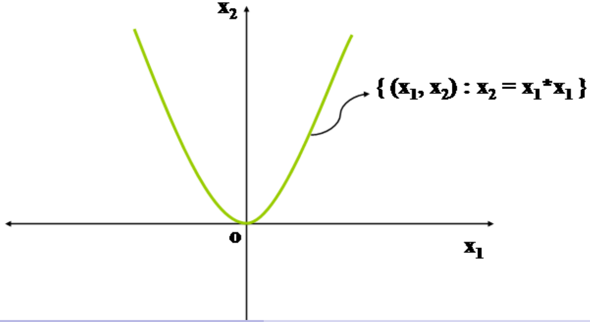
- Some simple convex sets



Some examples of nonconvex sets







Convex sets

Examples:

- The empty set ϕ, any singleton set $\{\mathbf{x}_0\}$ and $\mathbb{R}^n$ are convex subsets of $\mathbb{R}^n$.
- A closed ball in $\mathbb{R}^n$, $B[\mathbf{x}_0, r]$ and an open ball in $\mathbb{R}^n$, $B(\mathbf{x}_0, r)$, are convex sets.
- Any affine set is a convex set.
- A line segment is a convex set, but not an affine set.

- Possible to construct new convex sets using given convex sets

Definition

Let $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

- The *scalar multiple* αP of the set P is defined as

$$\alpha P = \{\mathbf{x} : \mathbf{x} = \alpha \mathbf{p}, \mathbf{p} \in P\}$$

- The *sum* of two sets P and Q is the set,

$$P + Q = \{\mathbf{x} : \mathbf{x} = \mathbf{p} + \mathbf{q}, \mathbf{p} \in P, \mathbf{q} \in Q\}.$$

Theorem

If $\{C_i\}, i \in \mathcal{A}$, is any collection of convex sets, then $\cap_{i\in\mathcal{A}} C_i$ is a convex set.

Proof.

Let $C = \cap_{i\in\mathcal{A}} C_i$.
Case 1. If C is *empty* or *singleton*, then clearly C is a convex set.
Case 2. To show that, for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\lambda \in [0, 1]$,

$$\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C.$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in C$. Clearly, $\mathbf{x}_1, \mathbf{x}_2 \in C_i$ for every $i \in \mathcal{A}$.
Since every C_i is convex,

$$\begin{aligned}
& \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C_i, \ \forall i \in \mathcal{A}, \lambda \in [0,1] \\
\Rightarrow\ & \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \cap_{i\in\mathcal{A}} C_i, \ \forall\ \lambda \in [0,1] \\
\Rightarrow\ & \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C, \ \forall\ \lambda \in [0,1].
\end{aligned}$$

□

Theorem

If C_1 and C_2 are convex sets, then $C_1 + C_2$ is a convex set.

Proof.

Let $\mathbf{x}_1, \mathbf{y}_1 \in C_1$ and $\mathbf{x}_2, \mathbf{y}_2 \in C_2$.
So, $\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2 \in C_1 + C_2$. We need to show,
$\lambda(\mathbf{x}_1 + \mathbf{x}_2) + (1-\lambda)(\mathbf{y}_1 + \mathbf{y}_2) \in C_1 + C_2 \; \forall \lambda \in [0, 1]$.
Since C_1 and C_2 are convex,
$\mathbf{z}_1 = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{y}_1 \in C_1 \quad \forall \, \lambda \in [0, 1]$, and
$\mathbf{z}_2 = \lambda \mathbf{x}_2 + (1-\lambda)\mathbf{y}_2 \in C_2 \quad \forall \, \lambda \in [0, 1]$.
Thus, $\mathbf{z}_1 + \mathbf{z}_2 \in C_1 + C_2 \ldots\ldots (1)$
Now, $\mathbf{x}_1 + \mathbf{x}_2 \in C_1 + C_2$ and $\mathbf{y}_1 + \mathbf{y}_2 \in C_1 + C_2$.
Therefore, from (1),
$\mathbf{z}_1 + \mathbf{z}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2) + (1-\lambda)(\mathbf{y}_1 + \mathbf{y}_2) \in C_1 + C_2 \; \forall \lambda \in [0, 1]$.
Thus, $C_1 + C_2$ is a convex set. $\square$

Theorem

If C is a convex set and $\alpha \in \mathbb{R}$, then αC is a convex set.

Proof.

Let $\mathbf{x}_1, \mathbf{x}_2 \in C$.
Since C is convex, $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in C \; \forall \; \lambda \in [0, 1]$.
Also, $\alpha \mathbf{x}_1, \alpha \mathbf{x}_2 \in \alpha C$.
Therefore, $\lambda(\alpha \mathbf{x}_1) + (1 - \lambda)(\alpha \mathbf{x}_2) = \alpha(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \in \alpha C$ for any $\lambda \in [0, 1]$.
Therefore, αC is a convex set. $\square$

Hyperplane

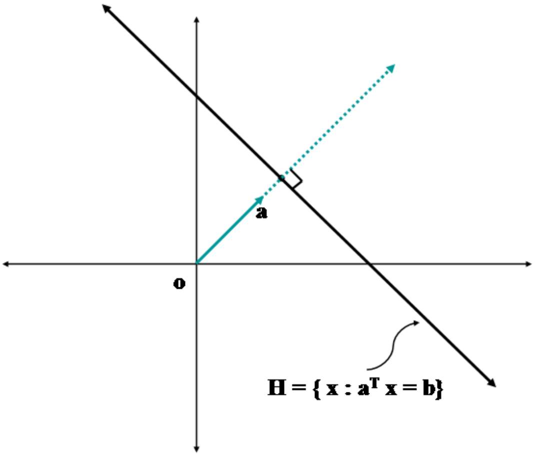
Definition

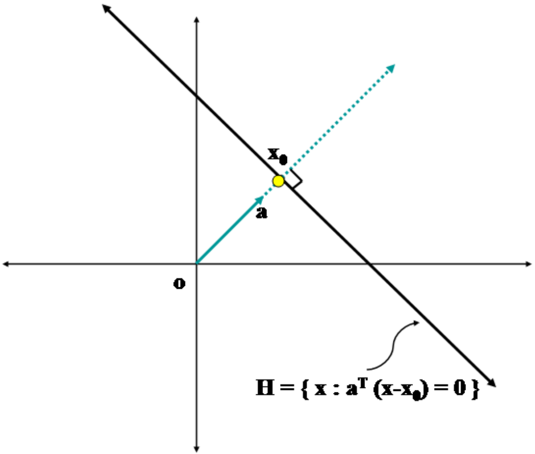
Let $b \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq 0$. Then, the set

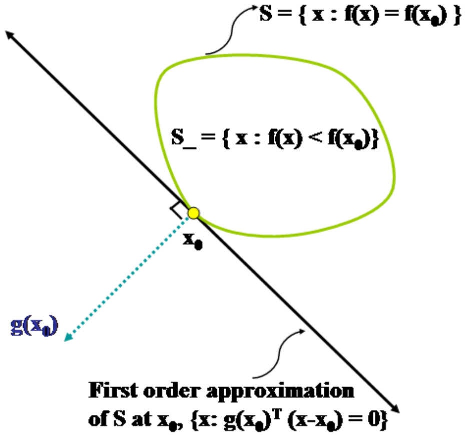
$$H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$$

is said to be a *hyperplane* in $\mathbb{R}^n$.

- $\mathbf{a}$ denotes the normal to the hyperplane H.
- If $\|\mathbf{a}\| = 1$, then $|b|$ is the distance of H from the origin.
- In $\mathbb{R}^2$, hyperplane is a line
- In $\mathbb{R}^3$, hyperplane is a plane

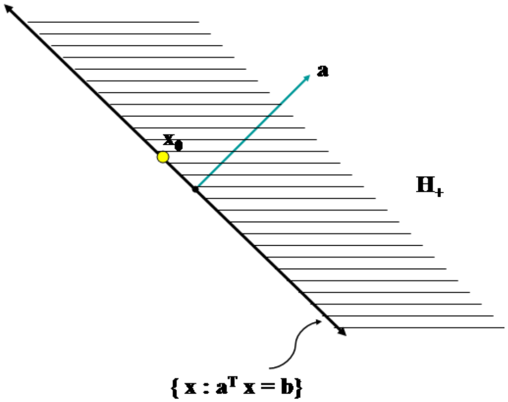

a
o
H = { x : a^T x = b}

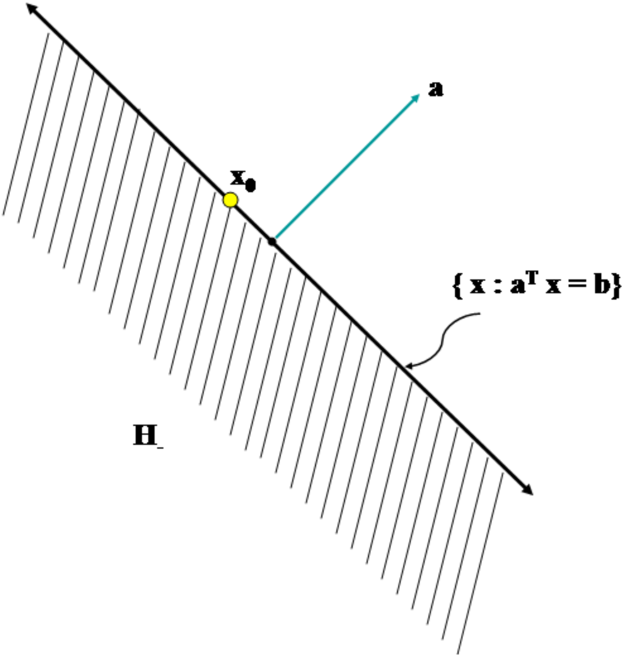

x0
a
o
H = { x : aT (x-x0) = 0 }


$S = \{ x : f(x) = f(x_0) \}$
$S\_ = \{ x : f(x) < f(x_0) \}$
x_0
$g(x_0)$
First order approximation
of S at x_0, $\{x: g(x_0)^T (x-x_0) = 0\}$

Half-spaces

- The sets $H_+ = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \geq b\}$ and $H_- = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \leq b\}$ are called closed positive and negative *half-spaces* generated by H.



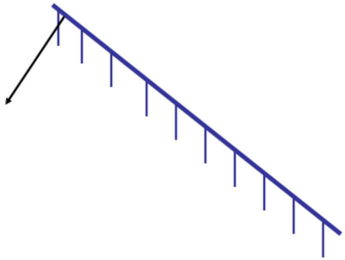

a
$\mathbf{x_0}$
$\{\mathbf{x} : \mathbf{a}^T \mathbf{x} = \mathbf{b}\}$
$\mathbf{H_-}$

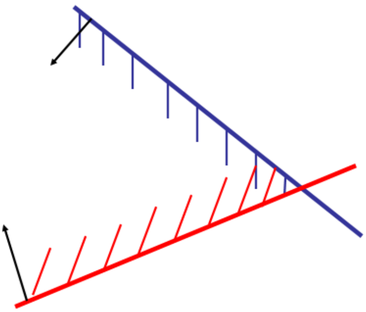
Some more examples of convex sets

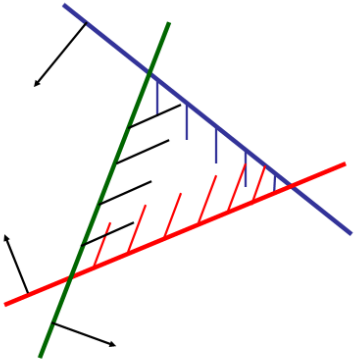
- $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$ is a convex set
- Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$

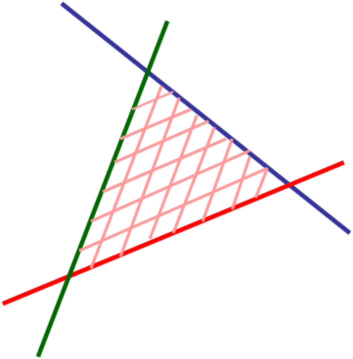
  Define, $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

  Then, $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is a convex set.
- $H_+ = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \geq b\}$ and $H_- = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \leq b\}$ are convex sets.
- $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ and $\{\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ are convex sets.









Convex Hull

Definition

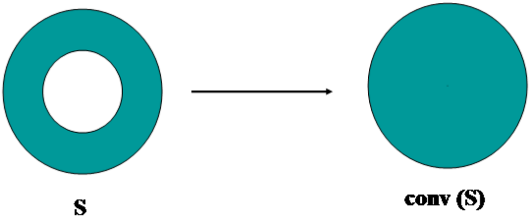
The *convex hull* of a set S is the intersection of all the convex sets which contain S and is denoted by $\text{conv}(S)$.

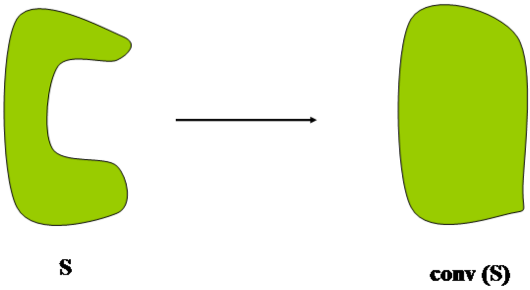
Note: Convex Hull of a set S is the convex set.

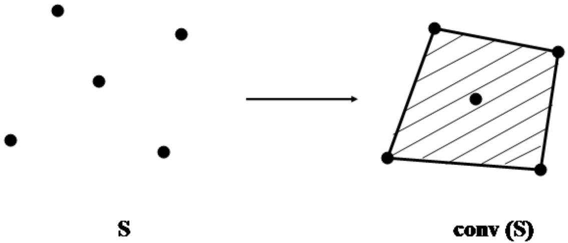
- The smallest convex set that contains S is called the *convex hull* of the set S.

Examples:

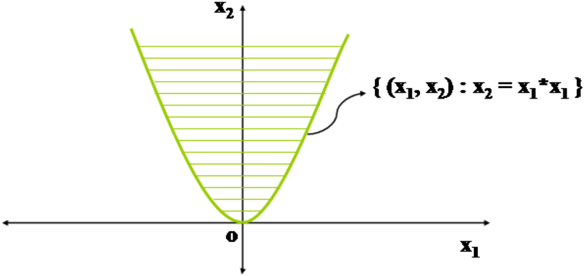
- $\text{conv}(\{x, y\}) = LS[x, y]$ where x and y are two points
- Let $S = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$. Then $\text{conv}(S) = \mathbb{R}^2$.


S
conv (S)


S
conv (S)



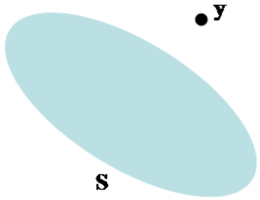
S
conv (S)

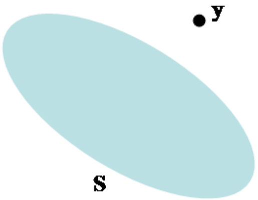


- $S = \{(x_1, x_2) : x_2 = x_1^2\}$
- $\text{conv}(S) = \{(x_1, x_2) : x_2 \geq x_1^2\}$

Theorem CS1

Let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\mathbf{y} \notin S$. Then there exists a unique point $\mathbf{x}_0 \in S$ with minimum distance from $\mathbf{y}$. Further, $\mathbf{x}_0$ is the minimizing point iff $(\mathbf{y} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq 0$ for all $\mathbf{x} \in S$.

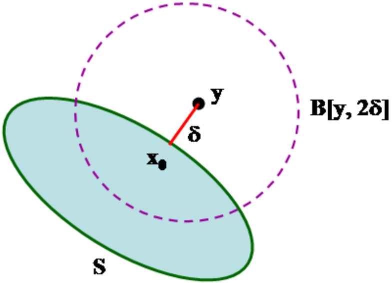




Proof Sketch

Given $S \subset \mathbb{R}^n$, a closed convex set and $\mathbf{y} \notin S$.

- First show that there exists a minimizing point $\mathbf{x}_0$ in S that is closest to $\mathbf{y}$.
- Let $\delta = \inf_{\mathbf{x} \in S} \|\mathbf{x} - \mathbf{y}\|$.
- Note that S is not bounded.



Proof Sketch (continued)

- Consider the set $S \cap B[\mathbf{y}, 2\delta]$.
- By Weierstrass' Theorem, there exists a minimizing point $\mathbf{x}_0$ in S that is closest to $\mathbf{y}$.

$$\mathbf{x}_0 = \arg\min_{\mathbf{x} \in S \cap B[\mathbf{y}, 2\delta]} ||\mathbf{x} - \mathbf{y}||$$

Proof Sketch (continued)

- Show uniqueness using triangle inequality
  Assume that there exists $\hat{\mathbf{x}} \in S$ such that

$$\|\mathbf{y} - \mathbf{x}_0\| = \|\mathbf{y} - \hat{\mathbf{x}}\| = \delta.$$

  Since S is convex, $\frac{1}{2}(\mathbf{x}_0 + \hat{\mathbf{x}}) \in S$.
  Using triangle inequality,

$$2\|\mathbf{y} - \frac{(\mathbf{x}_0 + \hat{\mathbf{x}})}{2}\| \leq \|\mathbf{y} - \mathbf{x}_0\| + \|\mathbf{y} - \hat{\mathbf{x}}\| = 2\delta$$

  We have a contradiction if strict inequality holds.

Proof Sketch (continued)

- To show that $\mathbf{x}_0$ is the unique minimizing point iff $(\mathbf{y}-\mathbf{x}_0)^T(\mathbf{x}-\mathbf{x}_0) \leq 0$ for all $\mathbf{x} \in S$.
  Let $\mathbf{x} \in S$. Assume $(\mathbf{y}-\mathbf{x}_0)^T(\mathbf{x}-\mathbf{x}_0) \leq 0$.

$$\begin{aligned}\|\mathbf{y}-\mathbf{x}\|^2 &= \|\mathbf{y}-\mathbf{x}_0+\mathbf{x}_0-\mathbf{x}\|^2 \\ &= \|\mathbf{y}-\mathbf{x}_0\|^2 + \|\mathbf{x}_0-\mathbf{x}\|^2 + 2(\mathbf{y}-\mathbf{x}_0)^T(\mathbf{x}_0-\mathbf{x})\end{aligned}$$

- Using the assumption, $(\mathbf{y}-\mathbf{x}_0)^T(\mathbf{x}_0-\mathbf{x}) \geq 0$, we get

$$\|\mathbf{y}-\mathbf{x}\|^2 \geq \|\mathbf{y}-\mathbf{x}_0\|^2$$

  This implies that $\mathbf{x}_0$ is the minimizing point.

Proof Sketch (continued)

- Assume that $\mathbf{x}_0$ is the minimizing point, that is,

$$\|\mathbf{y} - \mathbf{x}_0\|^2 \leq \|\mathbf{y} - \mathbf{z}\|^2 \; \forall \; \mathbf{z} \in S.$$

Consider any $\mathbf{x} \in S$. Since S is convex,

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}_0 \in S \; \forall \lambda \in [0, 1].$$

Therefore, $\|\mathbf{y} - \mathbf{x}_0\|^2 \leq \|\mathbf{y} - \mathbf{x}_0 - \lambda(\mathbf{x} - \mathbf{x}_0)\|^2$

That is,

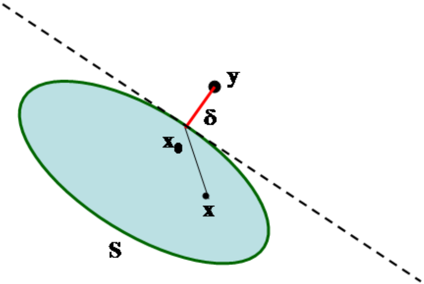
$\|\mathbf{y} - \mathbf{x}_0\|^2 \leq \|\mathbf{y} - \mathbf{x}_0\|^2 + \lambda^2 \|\mathbf{x} - \mathbf{x}_0)\|^2 - 2\lambda(\mathbf{y} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0)$

$2(\mathbf{y} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq \lambda \|\mathbf{x} - \mathbf{x}_0)\|^2$

Letting $\lambda \rightarrow 0^+$, the result follows. $\square$

Theorem CS1

Let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\mathbf{y} \notin S$. Then there exists a unique point $\mathbf{x}_0 \in S$ with minimum distance from $\mathbf{y}$. Further, $\mathbf{x}_0$ is the minimizing point iff $(\mathbf{y} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq 0$ for all $\mathbf{x} \in S$.

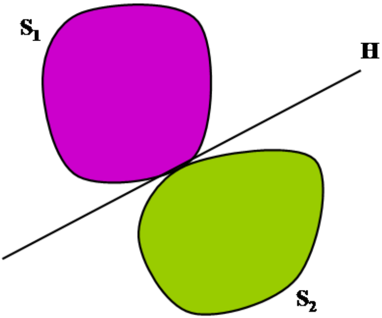


Separating hyperplanes

Definition

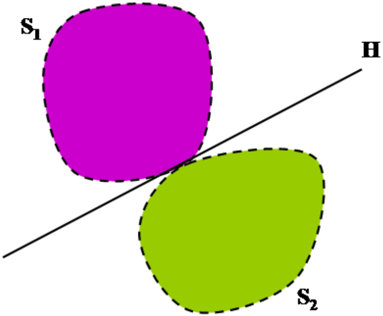
Let S_1 and S_2 be nonempty subsets in $\mathbb{R}^n$ and $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$ be a hyperplane.

1. The hyperplane H is said to *separate* S_1 and S_2, if $\mathbf{a}^T\mathbf{x} \geq b \;\forall\; \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} \leq b \;\forall\; \mathbf{x} \in S_2$.
2. If $\mathbf{a}^T\mathbf{x} > b \;\forall\; \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} < b \;\forall\; \mathbf{x} \in S_2$, then the hyperplane H is said to *strictly separate* S_1 and S_2.
3. The hyperplane H is said to *strongly separate* S_1 and S_2 if $\mathbf{a}^T\mathbf{x} \geq b + \epsilon \;\forall\; \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} \leq b \;\forall\; \mathbf{x} \in S_2$ where ϵ is a positive scalar.



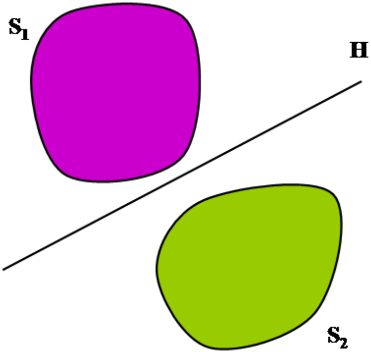
H: Separating hyperplane

The hyperplane $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$ is said to *separate* S_1 and S_2, if $\mathbf{a}^T\mathbf{x} \geq b \; \forall \; \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} \leq b \; \forall \; \mathbf{x} \in S_2$.



H: Strictly separating hyperplane

If $\mathbf{a}^T\mathbf{x} > b \ \forall \ \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} < b \ \forall \ \mathbf{x} \in S_2$, then the hyperplane $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$ is said to *strictly separate* S_1 and S_2.



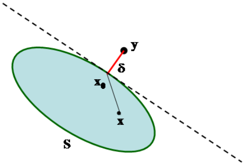
H: Strongly separating hyperplane

The hyperplane $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$ is said to *strongly separate* S_1 and S_2 if $\mathbf{a}^T\mathbf{x} \geq b + \epsilon \ \forall \ \mathbf{x} \in S_1$ and $\mathbf{a}^T\mathbf{x} \leq b \ \forall \ \mathbf{x} \in S_2$ where ϵ is a positive scalar.

Separation of a closed convex set and a point

Result

Let S be a nonempty closed convex set in $\mathbb{R}^n$ and $\mathbf{y} \notin S$. Then there exists a nonzero vector $\mathbf{a}$ and a scalar b such that $\mathbf{a}^T\mathbf{y} > b$ and $\mathbf{a}^T\mathbf{x} \le b \ \forall \ \mathbf{x} \in S$.



Proof.

By Theorem CS1, there exists a unique minimizing point $\mathbf{x}_0 \in S$ such that $(\mathbf{x} - \mathbf{x}_0)^T(\mathbf{y} - \mathbf{x}_0) \le 0$ for each $\mathbf{x} \in S$.
Letting $\mathbf{a} = (\mathbf{y} - \mathbf{x}_0)$ and $b = \mathbf{a}^T\mathbf{x}_0$, we get $\mathbf{a}^T\mathbf{x} \le b$ for each $\mathbf{x} \in S$ and $\mathbf{a}^T\mathbf{y} - b = (\mathbf{y} - \mathbf{x}_0)^T(\mathbf{y} - \mathbf{x}_0) > 0$ (since $\mathbf{y} \neq \mathbf{x}_0$). $\square$

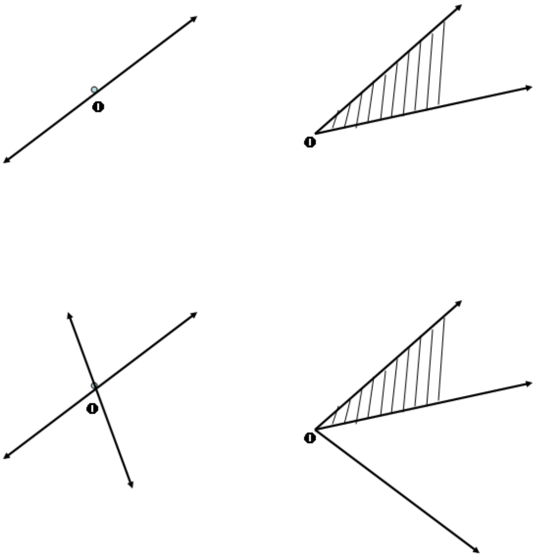
Cone

Definition

A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for every $\mathbf{x} \in K$ and $\lambda \geq 0$, we have $\lambda\mathbf{x} \in K$.

- K is a convex cone if it is convex and a cone.

Some examples of cones



Farkas' Lemma

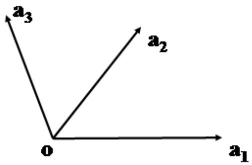
Farkas' Lemma

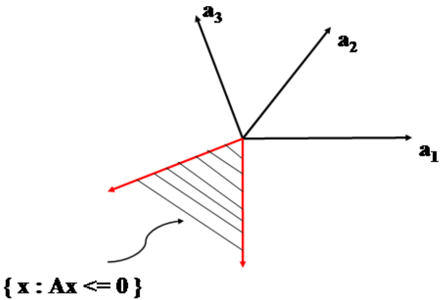
Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Then, exactly one of the following two systems has a solution:

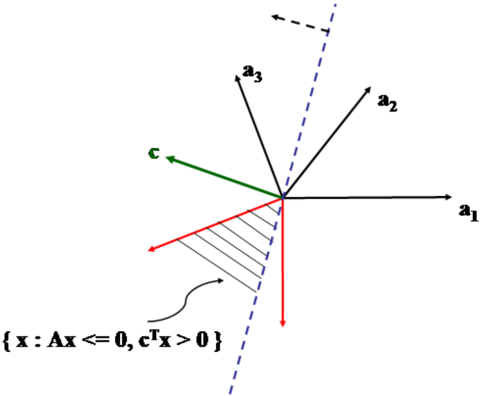
(I) $\mathbf{A}\mathbf{x} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0 \quad \text{for some } \mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0} \quad \text{for some } \mathbf{y} \in \mathbb{R}^m.$

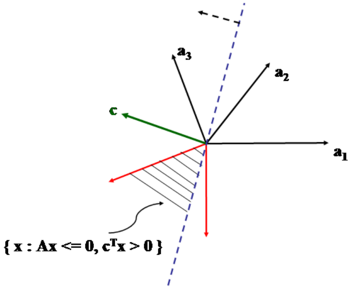
- Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$.
- Define, $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix}$.
- For example,

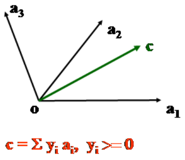



a3
a2
a1
{ x : Ax <= 0 }


a3
a2
c
a1
{ x : Ax <= 0, cTx > 0 }

Farkas' Lemma (Geometrical Interpretation)





Exactly one of the following two systems has a solution:

(I) $\mathbf{Ax} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$

Farkas' Lemma

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Then, exactly one of the following two systems has a solution:

(I) $\mathbf{Ax} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0 \quad$ for some $\mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0} \quad$ for some $\mathbf{y} \in \mathbb{R}^m$

Proof of Farkas' Lemma

(a) Suppose, system (II) has a solution.

Therefore, $\exists \mathbf{y} \geq \mathbf{0}$ such that $\mathbf{c} = \mathbf{A}^T\mathbf{y}$.
Consider $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} \leq \mathbf{0}$.
So, $\mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{Ax} \leq 0$.
That is, System I has no solution.

Farkas' Lemma

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Then, exactly one of the following two systems has a solution:

(I) $\mathbf{A}\mathbf{x} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0 \quad \text{for some } \mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0} \quad \text{for some } \mathbf{y} \in \mathbb{R}^m$

Proof of Farkas' Lemma (continued)

(b) Suppose, system (II) has no solution.
Let $S = \{\mathbf{x} : \mathbf{x} = \mathbf{A}^T\mathbf{y}, \mathbf{y} \geq 0\}$, a closed convex set and $\mathbf{c} \notin S$.
Therefore, $\exists \mathbf{p} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{p}^T\mathbf{x} \leq \alpha \ \forall \ \mathbf{x} \in S \text{ and } \mathbf{c}^T\mathbf{p} > \alpha.$$

This means, $\alpha \geq 0$ (Since $\mathbf{0} \in S$) $\Rightarrow \mathbf{c}^T\mathbf{p} > 0$.
Also, $\alpha \geq \mathbf{p}^T\mathbf{A}^T\mathbf{y} = \mathbf{y}^T\mathbf{A}\mathbf{p} \ \forall \ \mathbf{y} \geq \mathbf{0}$.
Since $\mathbf{y} \geq \mathbf{0}$, $\mathbf{A}\mathbf{p} \leq \mathbf{0}$ (as $\mathbf{y}$ can be made arbitrarily large).
Thus, $\exists \mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{p} \leq \mathbf{0}, \mathbf{c}^T\mathbf{p} > 0 \Rightarrow$ System (I) has a solution.

Farkas' Lemma

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Then, exactly one of the following two systems has a solution:

(I) $\mathbf{A}\mathbf{x} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0 \quad$ for some $\mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0} \quad$ for some $\mathbf{y} \in \mathbb{R}^m$.

Corollary

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the following systems has a solution:

(I) $\mathbf{A}\mathbf{x} < \mathbf{0} \quad$ for some $\mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{0}, \mathbf{y} \geq \mathbf{0} \quad$ for some nonzero $\mathbf{y} \in \mathbb{R}^m$.

Corollary

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the following systems has a solution:

(I) $\mathbf{A}\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$

(II) $\mathbf{A}^T\mathbf{y} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}$ for some nonzero $\mathbf{y} \in \mathbb{R}^m$.

Proof.

We can write system I as

$$\mathbf{A}\mathbf{x} + z\mathbf{e} \leq \mathbf{0} \text{ for some } \mathbf{x} \in \mathbb{R}^n, z > 0$$

where $\mathbf{e}$ is a m-dimensional vector containing all 1's.
That is,

$$(\mathbf{A} \quad \mathbf{e}) \begin{pmatrix} \mathbf{x} \\ z \end{pmatrix} \leq \mathbf{0}, \quad (0, \ldots, 0, 1) \begin{pmatrix} \mathbf{x} \\ z \end{pmatrix} > 0$$

for some $(\mathbf{x} \quad z)^T \in \mathbb{R}^{n+1}$.

Proof. (continued)

Recall Farkas' Lemma.
Therefore, System II is

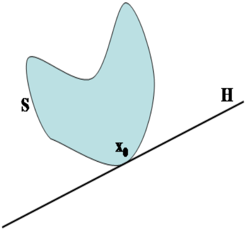
$$\begin{pmatrix} \mathbf{A}^T \\ \mathbf{e}^T \end{pmatrix} \mathbf{y} = (0, \ldots, 0, 1)^T, \mathbf{y} \geq \mathbf{0} \text{ for some } \mathbf{y} \in \mathbb{R}^m.$$

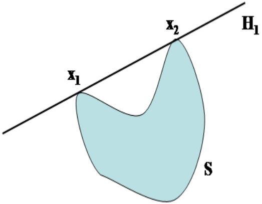
That is $\mathbf{A}^T\mathbf{y} = 0, \mathbf{e}^T\mathbf{y} = 1$ and $\mathbf{y} \geq 0$ for some $\mathbf{y} \in \mathbb{R}^m$.
Using Farkas' Lemma, the result follows.

Supporting Hyperplanes of sets at boundary points

Definition

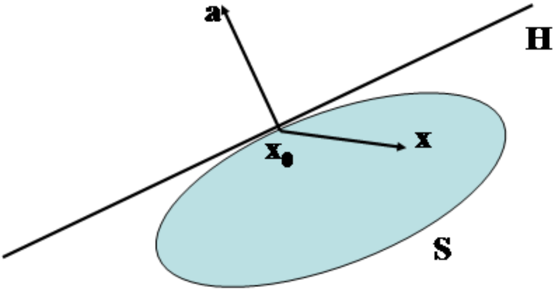
Let $S \subset \mathbb{R}^n$, $S \neq \phi$. Let $\mathbf{x}_0$ be a boundary point of S. A hyperplane $H = \{\mathbf{x} : \mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) = 0\}$ is called a *supporting hyperplane* of S at $\mathbf{x}_0$, if either
$S \subseteq H_+$ (that is, $\mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) \geq 0 \; \forall \; \mathbf{x} \in S$), or
$S \subseteq H_-$ (that is, $\mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) \leq 0 \; \forall \; \mathbf{x} \in S$).





Theorem

Let S be a nonempty convex set in $\mathbb{R}^n$ and $\mathbf{x}_0$ be a boundary point of S. Then, there exists a hyperplane that supports S at $\mathbf{x}_0$.



Theorem

Let S_1 and S_2 be two nonempty disjoint convex sets in $\mathbb{R}^n$. Then, there exists a hyperplane that separates S_1 and S_2.

Proof.

Consider the set $S = S_1 - S_2 = \{\mathbf{x}_1 - \mathbf{x}_2 : \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$.
Note that S is a convex set and $\mathbf{0} \notin S$.
Thus, we have a convex set and a point not in that set.
So, there exists a vector $\mathbf{a}$ such that $\mathbf{a}^T\mathbf{x} \leq 0 \ \forall\ \mathbf{x} \in S$.
In other words, $\mathbf{a}^T\mathbf{x}_1 \leq \mathbf{a}^T\mathbf{x}_2 \ \forall\ \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2$.

$\square$