1 Contract an edge.
2 Delete a vertex.
3 Delete an edge.
Let $G$ be a graph. If $X$ is another graph, and \( \{ V_x : x \in V(X) \} \) is a partition of $V(G)$ into connected subsets such that for any two vertices $x, y \in X$, there is a $V_x$-$V_y$ edge in $G$ if and only if $(x, y) \in E(G)$, we say that $G$ is an $MX$ and write $G = MX$. $V_x$ are the branch sets of $MX$. 
\( G \) is an MX if and only if \( X \) can be obtained from \( G \) by a series of edge contractions, i.e. if and only if there are graphs \( G_0, \ldots, G_n \) and edges \( e_i \in G_i \) such that \( G_0 = G \), \( G_n = X \) and \( G_{i+1} = G_i / e_i \), for all \( i < n \).
If $G = MX$ is a subgraph of another graph $Y$, then we say that $X$ is a minor of $Y$. 
If we replace the edges of $X$ with independent paths between ends, we call the graph $G$ obtained a subdivision of $X$, and write $G = TX$. If $TX$ is a subgraph of $Y$ then $X$ is a topological minor of $Y$. 
If $\Delta(X) \leq 3$ then every $MX$ contains a $TX$. 
Hadwiger’s Conjecture: The following implication holds for every integer $r > 0$ and every graph $G$.

$\chi(G) \geq r$ implies that $K_r$ is a minor of $G$. 

A graph with at least 3 vertices is edge maximal without a $K_4$ minor if and only if it can be constructed recursively from triangles by pasting along $K_2$s.
Every edge maximal graph without a $K_4$ minor has $2|G| - 3$ edges.
Hadwiger’s Conjecture holds for $r = 4$
Wagner, 1937: Let $G$ be an edge maximal graph without a $K_5$ minor. If $|G| \geq 4$, then $G$ can be constructed recursively, by pasting along $K_3$s and $K_2$s from plane triangulations and copies of the graph $W$. 
A graph with $n$ vertices and no $K_5$ minor has at most $3n - 6$ edges.
Hadwiger’s conjecture holds for $r = 5$. 
(Robertson, Seymour and Thomas, 1993) Hadwiger’s conjecture holds for $r = 6$. 
Kühn and Osthus: For every integer $s$, there is an integer $r_s$ such that Hadwiger Conjecture holds for all graphs $G \nsubseteq K_{s,s}$ and $r \geq r_s$. 
There is a constant $g$ such that all graphs $G$ of girth at least $g$ satisfy the implication
\[\chi(G) \geq r \rightarrow G \supseteq TK_r \text{ for all } r.\]
There is a constant $c \in \mathbb{R}$ such that for $r \in \mathbb{N}$, every graph $G$ of average degree $d(G) \geq cr^2$ contains $K_r$ as a topological minor.
Kostochka, 1982: There exists a constant $c \in \mathbb{R}$ such that for every $r \in \mathbb{N}$, every graph $G$ of average degree $d(G) \geq cr \sqrt{\log r}$ contains $K_r$ as a minor. Up to the value of $c$, this bound is best possible as a function of $r$. 
Let $d, k \in \mathbb{N}$ with $d \geq 3$ and let $G$ be a graph of minimum degree $\delta(G) \geq d$ and girth $g(G) \geq 8k + 3$. Then $G$ has a minor $H$ of minimum degree $\delta(H) \geq d(d - 1)^k$. 
Thomassen, 1983: There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$, such that every graph of minimum degree at least 3 and girth at least $f(r)$ has a $K_r$ minor, for all $r \in \mathbb{N}$. 
Take $f(r) = 8 \log r + 4 \log \log r + c$ for some constant $c \in \mathbb{R}$. Take $k = k(r)$ minimal with $3.2^k \geq c' r \sqrt{\log r}$, where $c'$ is the constant from Kostochka’s Lemma.
There exists a constant $g$ such that $G \supseteq TK_r$ for every graph $G$ satisfying $\delta(G) \geq r - 1$ and girth $\geq g$. 