Let $k > 0$ be an integer, and let $p = p(n)$ be a function of $n$ such that $p \geq \frac{6k \ln n}{n}$ for large $n$. Then $\lim_{n \to \infty} P(\alpha \geq \frac{n}{2k}) = 0$
For every integer $k$, there exists a graph $H$ with girth $g(H) > k$ and chromatic number $\chi(H) > k$. 
- \( k \geq 3 \), and fix \( 0 < \epsilon < \frac{1}{k} \). Let \( p = n^{\epsilon-1} \)
- \( \text{Let } X(G) \text{ denote the number of short cycles (of length at most } k \text{) in a random graph } G \in \mathcal{G}(n, p). \)
- \( E(X) = \sum_{i=3}^{k} \frac{(n)^i p^i}{2i} \leq \frac{1}{2} \sum_{i=3}^{k} (n^i p^i) \leq \frac{1}{2}(k - 2)n^k p^k. \)
- \( P[X \geq n/2] \leq \frac{E(X)}{n/2} \leq (k - 2)n^{k-1} p^k = (k - 2)n^{k\epsilon-1}. \)
- Since \( k\epsilon - 1 < 0 \), \( \lim_{n \to \infty} P[X \geq n/2] = 0. \)
Let $\mathcal{P}$ be a graph property - i.e. a class of graphs closed under isomorphism.

Let $p = p(n)$ be a fixed function. If $P[G \in \mathcal{P}] \to 1$, as $n \to \infty$, we say that $G \in \mathcal{P}$ for almost all $G \in \mathcal{G}(n, p)$.

If $P[G \in \mathcal{P}] \to 0$ as $n \to \infty$, we say that almost no $G \in \mathcal{G}(n, p)$ has property $\mathcal{P}$. 
For every constant $p \in (0, 1)$, and every graph $H$, almost every $G \in G(n, p)$, contains an induced copy of $H$. 
We call a real function $t = t(n)$ with $t(n) \neq 0$, for all $n$, a threshold function for a graph property $\mathcal{P}$, if the following holds for all $p = p(n)$, and $G \in G(n, p)$.

$$\lim_{n \to \infty} [G \in \mathcal{P}] = 0, \text{ if } p/t \to 0, \text{ as } n \to \infty \text{ and } \lim_{n \to \infty} [G \in \mathcal{P}] = 1 \text{ if } p/t \to 1, \text{ as } n \to \infty.$$
Consider a graph property of the form $\mathcal{P} = \{ G : X(G) \geq 1 \}$ where $X \geq 0$ is a random variable on $G(n, p)$. (Example, connectedness).

How can we prove that $\mathcal{P}$ has a threshold function $t$?

We study one method here, called second moment method.

If we can show that as $n \to \infty$, $E(X) \to 0$, then it means, that almost all graphs have property $\mathcal{P}$. (Since $P[X \geq 1] \leq E(X)$, by Markov inequality.)

On the other hand we cannot show $P[X \geq 1] \to 1$ simply by showing a lower bound on $E(X)$ as $n \to \infty$. 
The Variance $\sigma^2$ of $X$: $\sigma^2 = E((X - \mu)^2)$. It is a quadratic measure of how much $X$ deviates from its mean.
$$\sigma^2 = E(X^2) - \mu^2.$$
Chebyshev’s Inequality: For all real $\lambda > 0$,  
\[ P[|X - \mu| \geq \lambda] \leq \frac{\sigma^2}{\lambda^2}. \]
If $\mu > 0$, for $n$ large, and $\frac{\sigma^2}{\mu^2} \to 0$, as $n \to \infty$, then $X(G) > 0$

Since any graph $G$ with $X(G) = 0$ satisfies $|X(G) - \mu| = \mu$. So,

$P[X = 0] \leq P[|X - \mu| \geq \mu] \leq \frac{\sigma^2}{\mu^2} \to 0$, as $n \to 0$. 

Given a graph $H$, let $\mathcal{P}_H$ be the property of containing a copy of $H$ as subgraph. $H$ is called balanced if $\epsilon(H') \leq \epsilon(H)$ for all subgraphs $H'$ of $H$. 
If $H$ is a balanced graph with $k$ vertices, and $\ell \geq 1$ edges, then $t(n) = n^{-k/\ell}$ is a threshold function for $\mathcal{P}_H$. 
If \( k \geq 3 \), then \( t(n) = n^{-1} \) is a threshold function for the property of containing a \( k \)-cycle.
If $T$ is a tree of order $k \geq 2$, then $t(n) = n^{k/(k-1)}$ is a threshold function for the property for containing a copy of $T$. 
If $k \geq 2$, then $t(n) = \frac{n^2}{(k-1)}$ is a threshold function for the property of containing a $K_k$. 
Let $X(G)$ denote the number of subgraphs of $G$ isomorphic to $H$.

Given $n \in \mathbb{N}$, let $\mathcal{H}$ denote the set of all graphs isomorphic to $H$ whose vertices lie in $\{0, 1, \ldots, n - 1\}$.

Given $H' \in \mathcal{H}$, we write $H' \subseteq G$ to denote that $H'$ itself is a subgraph of $G$.

The number of isomorphic copies of $H$ on a fixed $k$ set is at most $k!$.

$|\mathcal{H}| \leq \binom{n}{k} k! \leq n^k$.

Given $p = p(n)$, let $\gamma = p/t$, where $t = n^{-k/\ell}$. 
For each fixed $H' \in \mathcal{H}$, $P[H' \subseteq G] = p^\ell$ since $|E(H')| = \ell$.

$E(X) = |\mathcal{H}|p^\ell \leq n^k(\gamma n^{-k/\ell})^\ell = \gamma^\ell \to 0$, if $\gamma \to 0$ as $n \to 0$. 

We have \( \frac{n^k}{n^k} \geq \frac{1}{k!} \left( 1 - \frac{k-1}{k} \right)^k \).