Graph Theory: Lecture No. 34

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Let $D = (V, A)$ be a directed graph. Suppose that with each arc $a$ of $D$, are associated two real numbers $b(a)$ and $c(a)$ such that $b(a) \leq c(a)$. A circulation $f$ in $D$ is feasible (with respect to the functions $b$ and $c$) if $b(a) \leq f(a) \leq c(a)$ for all $a \in A$. A feasible tension is defined analogously.
An obvious necessary condition for the existence of a feasible circulation: For each subset $X$ of $V$, $c^+(X) \geq b^-(X)$
Hoffman’s Circulation Theorem: A digraph $D$ has a feasible circulation with respect to bounds $b$ and $c$ if and only if these bounds satisfy the above inequality for every subset $X$ of $V$. Furthermore, if both $b$ and $c$ are integral valued, and satisfy this inequality, then $D$ has an integer-valued feasible circulation.
Ghouila-Houri’s Theorem: A digraph $D$ has a feasible tension with respect to bounds $b$ and $c$ if and only if these bounds satisfy:

\[ b(C^-) \leq c(C^+) \]

for all cycles $C$ in $D$ with a sense of traversal. If $b$ and $c$ are integer valued and satisfy the above inequality, then $D$ has an integer valued feasible tension.
A function $f$ on the arc set $A$ of digraph $D$ is nowhere-zero if $f(a) \neq 0$ for each arc $a \in A$, i.e. if the support of $f$ is the entire arc set $A$. 
A nowhere zero circulation $f$ over $Z$ in a digraph $D$ is called a $k$-flow if 
$-(k - 1) \leq f(a) \leq (k - 1)$, for all $a \in A$. 

A graph admits a nowhere zero circulation over $\mathbb{Z}_k$ if and only if it admits a $k$-flow.
The flow number of a graph is defined to be the smallest positive integer $k$, for which it has a $k$-flow.
A graph admits a 2-flow if and only if it is even
A 2-edge connected cubic graph admits a 3-flow if and only if it is bipartite
Tutte’s Flow conjectures:
(1) The 5-flow conjecture: Every 2-edge connected graph admits a 5-flow.
(2) The 4-flow conjecture: Every 2-edge connected graph without a peterson graph minor admits a 4-flow.
(3) The 3-flow conjecture: Every 2-edge connected graph without 3-edge cuts admits a 3-flow.