Graph Theory: Lecture No. 2

L. Sunil Chandran

Computer Science and Automation,
Indian Institute of Science, Bangalore
Email: sunil@csa.iisc.ernet.in
A set \( M \) of independent edges in \( G \) is called a matching.

- Matched vertex
- Unmatched vertex
The cardinality of the biggest matching in $G$ can be denoted by $\alpha'(G)$. 
What is the value of $\alpha'(G)$ for:

- Cycle $C_n$
- Path $P_n$
- Complete Graph $K_n$
- Complete Bipartite graph $K_{m,n}$
If every vertex of $G$ is matched with respect to a matching $M$, then it is called a perfect matching.
How many edges are there in a perfect matching, if $G$ has $n$ vertices? What can we tell about $n$?
A perfect matching is also known as a 1-factor.

A $k$-factor is a $k$-regular spanning subgraph of $G$.

What can we tell about a 2-factor.
In general do we have any relation between $\alpha'(G)$ and $\alpha(G)$?

$\alpha(G) \geq n - 2\alpha'(G)$

So, do we have any relation between the minimum vertex cover and maximum matching?

$n - \beta(G) \geq n - 2\alpha'(G)$

$\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$
A stronger relation holds in bipartite graphs (König, 1931)

For a bipartite graph $G$, the maximum cardinality of a matching is equal to the minimum cardinality of its vertex cover
Suppose König’s statement, namely \( \beta(G) = \alpha'(G) \) is true, for bipartite graphs. And we are trying to come up with a proof.
Suppose $M$ is a matching such that $|M| = \alpha'(G)$.
For proving the theorem we will try to demonstrate a vertex cover $S$, with $|S| = \alpha'(G)$
$S$ should be such that it contains exactly one point from each edge of $M$
So we see that we are forced to add some edges in $S$. Let us try to understand this.
An alternating path: A path that starts at an unmatched vertex in $A$ and then contains alternately edges from $E - M$ and $M$. If an alternating path ends at an unmatched vertex, then it is called an augmenting path. An augmenting path starts from an unmatched vertex on the $A$ side, and ends at an unmatched vertex on the $B$ side. If we can find in $G$ and augmenting path with respect to $M$, then $M$ is not a maximum matching.
Hall’s Condition:
For all $S \subseteq A$, $|N(S)| \geq |S|$.
Hall’s Theorem

A bipartite graph $G$ has a matching of $A$ if and only if $G$ satisfies Hall’s condition.
Using Hall’s Theorem:

If $G$ is $k$-regular ($k \geq 1$) bipartite graph, then it has a perfect matching