Module 1: Pigeon Hole Principle

1. (a) Show that if \( n + 1 \) integers are chosen from the set \( \{1, 2, \ldots, 2n\} \) then there are always two which differ by 1.

   (b) Show that if \( n + 1 \) integers are chosen from the set \( \{1, 2, \ldots, 3n\} \) then there are always two which differ by at most 2.

   (c) Generalise the above 2 statements.

2. Use the pigeon hole principle to prove that the decimal expansion of a rational number \( m/n \) eventually is repeating.

3. Prove that any 5 points chosen within a square of side length 2, there are 2 whose distance apart is at most \( \sqrt{2} \).

4. There are 100 people at a party. Each person has an even number (possibly 0) of acquaintances. Prove that there are 3 people at the party with the same number of acquaintances.

5. Prove that in a graph of \( n \) vertices, where \( n \geq 6 \), there exists either a triangle (i.e. a complete subgraph on 3 vertices) or an independent set on 3 vertices.

Module 2: Elementary Concepts and Basic Counting Principles

1. Prove that the number of permutations of \( m \) A’s and at most \( n \) B’s equals

\[
\binom{m + n + 1}{m + 1}
\]

2. Consider the multiset \{n.a, 1, 2, \ldots, n\} of size 2n. Determine the number of its \( n \)-combinations.

3. Consider the multiset \{n.a, n.b, 1, 2, \ldots, n + 1\} of size 3n + 1. Determine the number of \( n \)-combinations.

4. Establish a bijection between the permutations of the set \{1, 2, \ldots, n\} and the towers of the form \( A_0 \subset A_1 \subset A_n \), where \( |A_k| = k \) for \( k = 0, 1, \ldots, n \).

5. A city has \( n \) junctions. It is decided that some of them will get traffic lights, and some of those that get traffic lights will also get a gas station. If at least one gas station comes up, then in how many different ways can this happen?
Module 3: More Strategies

1. Find the number of integers between 1 and 10,000 which are neither perfect squares nor perfect cubes.

2. Determine the number of 12-combinations of the multi-set \( S = \{4.a, 3.b, 4.c, 5.d\} \).

3. Determine a general formula for the number of permutations of the set \( \{1,2,\ldots,n\} \) in which exactly \( k \) integers are in their natural positions.

4. Use a combinatorial reasoning to derive the identity:
\[
n! = \sum_{i=0}^{n} \binom{n}{i} D_{n-i},
\]
where \( D_i \) is the number of permutations of \( \{1,2,\ldots,i\} \) such that no number is in its natural position. \( D_0 \) is defined to be 1.

5. Find the number of permutations of \( a, b, c, \ldots, x, y, z \) in which none of the patterns spin, game, path, and net occurs.

Module 4: Recurrence Relations and Generating Functions

1. Prove that the Fibonacci sequence is the solution of the recurrence relation
\[
a_n = 5a_{n-4} + 3a_{n-5}, \quad n \geq 5
\]
where \( a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2 \) and \( a_4 = 3 \). Then use this formula to show that the Fibonacci numbers satisfy the condition that \( f_n \) (the \( n \)-th Fibonacci number = \( a_n \)) is divisible by 5 if and only if \( n \) is divisible by 5.

2. Consider a 1-by-\( n \) chess board. Suppose we color each square of the chess board with one of 3 colors, red, green and blue so that no two squares that are colored red are adjacent. Let \( h_n \) be the number of such colorings possible. Find and verify a recurrence relation that \( h_n \) satisfies. Then find a formula for \( h_n \).

3. Solve the recurrence relation \( h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}, \quad n \geq 3 \) with initial values \( h_0 = 0, h_1 = 1, h_2 = 2 \).

4. Solve the non-homogenous recurrence relation \( h_n = 4h_{n-1} + 3.2^n, \quad n \geq 1 \) with initial value, \( h_0 = 1 \).

5. Let \( h_n \) be the number of ways to color the squares of a 1-by-\( n \) chess board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence \( h_0, h_1, h_2, \ldots \), and then find a simple formula for \( h_n \).

Module 5: Special Numbers

1. Let \( 2n \) (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting \( n \) line segments do not intersect, equals the \( n \)-th Catalan number.
2. Consider the Sterling Number of the second kind, \( S(n, k) \). Show that \( S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4} \).

3. The number of partitions of a set of \( n \) elements into \( k \) distinguishable boxes (some of which may be empty) is \( k^n \). By counting in a different way, prove that

\[
k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n, i)
\]

If \( k > n \), define \( S(n, k) = 0 \).

4. For each integer \( n > 2 \), determine a self-conjugate partition of \( n \) that has at least two parts.

5. Consider Sterling number of the first kind, \( s(n, k) \). Show that \( \sum_{i=0}^{n} s(n, i) = n! \)

6. Let \( t_1, t_2, \ldots, t_m \) be distinct positive integers, and let \( q_n = q_n(t_1, t_2, \ldots, t_n) \) equal the number of partitions of \( n \) in which all parts are taken from \( t_1, t_2, \ldots, t_m \). Define \( q_0 = 1 \). Show that the generating function for the sequence \( q_0, q_1, q_2, \ldots, \) is \( \prod_{k=1}^{m} \left( 1 - x^{t_k} \right)^{-1} \). 