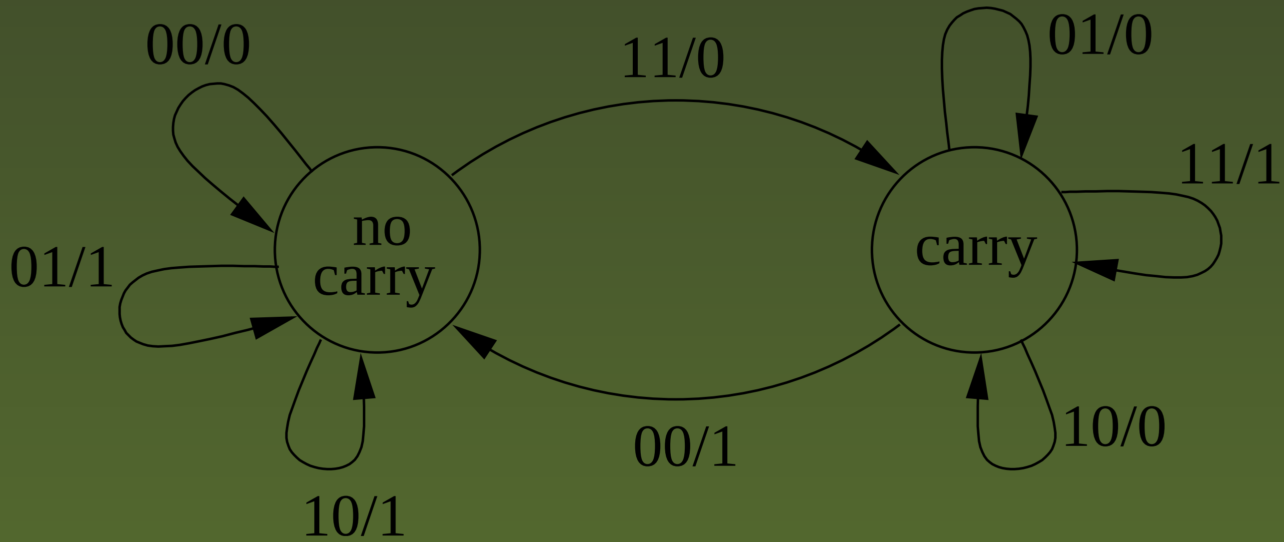
Introduction to Formal Languages, Automata and Computability

Finite State Automata

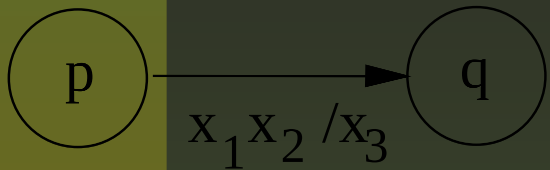
K. Krithivasan and R. Rama

Introduction

As another example consider a binary serial adder. At any time it gets two binary inputs x_1 and x_2. The adder can be in any one of the states ‘carry’ or ‘no carry’ . The four possibilities for the inputs $x_1 x_2$ are $00, 01, 10, 11$. Initially the adder is in the ‘no carry’ state. The working of the serial adder can be represented by the following diagram.



contd.



denotes that when the adder is in state p and gets input x_1x_2, it goes to state q and outputs x_3. The input and output on a transition from p to q is denoted by i/o. It can be seen that suppose the two binary numbers to be added are 100101 and 100111.

Time	6	5	4	3	2	1
	1	0	0	1	0	1
	1	0	0	1	1	1

The input at time $t = 1$ is 11 and the output is 0 and the

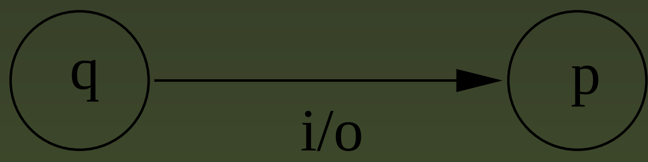
contd

machine goes to ‘carry’ state. The output is 0. Here at time $t = 2$, the input is 01; the output is 0 and the machine remains in ‘carry’ state. At time $t = 3$, it gets 11 and output 1 and remains in
‘carry’ state. At time $t = 4$, the input is 00; the machine outputs 1 and goes to ‘no carry’ state. At time $t = 5$, the input is 00; the output is 0 and the machine remains in ‘no carry’ state. At time $t = 6$, the input is 11; the machine outputs 0 and goes to ‘carry’ state. The input stops here. At time $t = 7$, no input is there (and this is taken as 00) and the output is 1.

It should be noted that at time $t = 1, 3, 6$ input is 11, but the output is 0 at $t = 1, 6$ and is 1 at time $t = 3$. At time $t = 4, 5$, the input is 00, but the output is 1 at time $t = 4$ and 0 at $t = 5$.

contd.

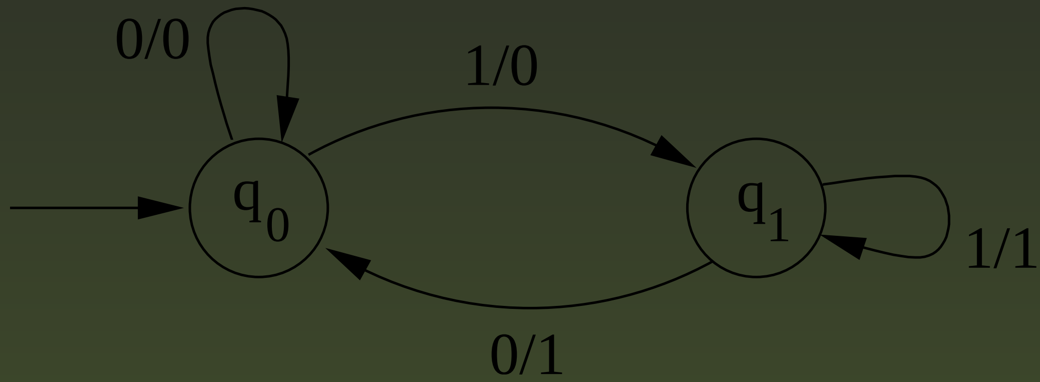
So it is seen that the output depends both on the input and the state.
The diagrams we have seen are called state diagrams.



The above diagram indicates that in state q when the machine gets input i, it goes to state p and outputs 0.

Let us consider one more example of a state diagram given in

contd



The input and output alphabet are $\{0, 1\}$. For the input 011010011, the output is 00110100 and machine is in state q_1. It can be seen that the first is 0 and afterwards, the output is the symbol read at the previous instant. It can also be noted that the machine goes to q_1 after reading a 1 and goes to q_0 after reading a 0.

contd

It should also be noted that when it goes from state q_0 it outputs a 0 and when it goes from state q_1, it outputs a 1. This machine is called a one moment delay machine.

Deterministic Finite State Automaton

Definition A Deterministic Finite State Automaton (DFSA) is a 5-tuple
$M = (K, \Sigma, \delta, q_0, F)$ where

- K is a finite set of states
- Σ is a finite set of input symbols
- q_0 in K is the start state or initial state
- $F \subseteq K$ is set of final states
- δ, the transition function is a mapping from $K \times \Sigma \rightarrow K$.

$\delta(q, a) = p$ means, if the automaton is in state q and reading a symbol a, it goes to state p in the next

contd

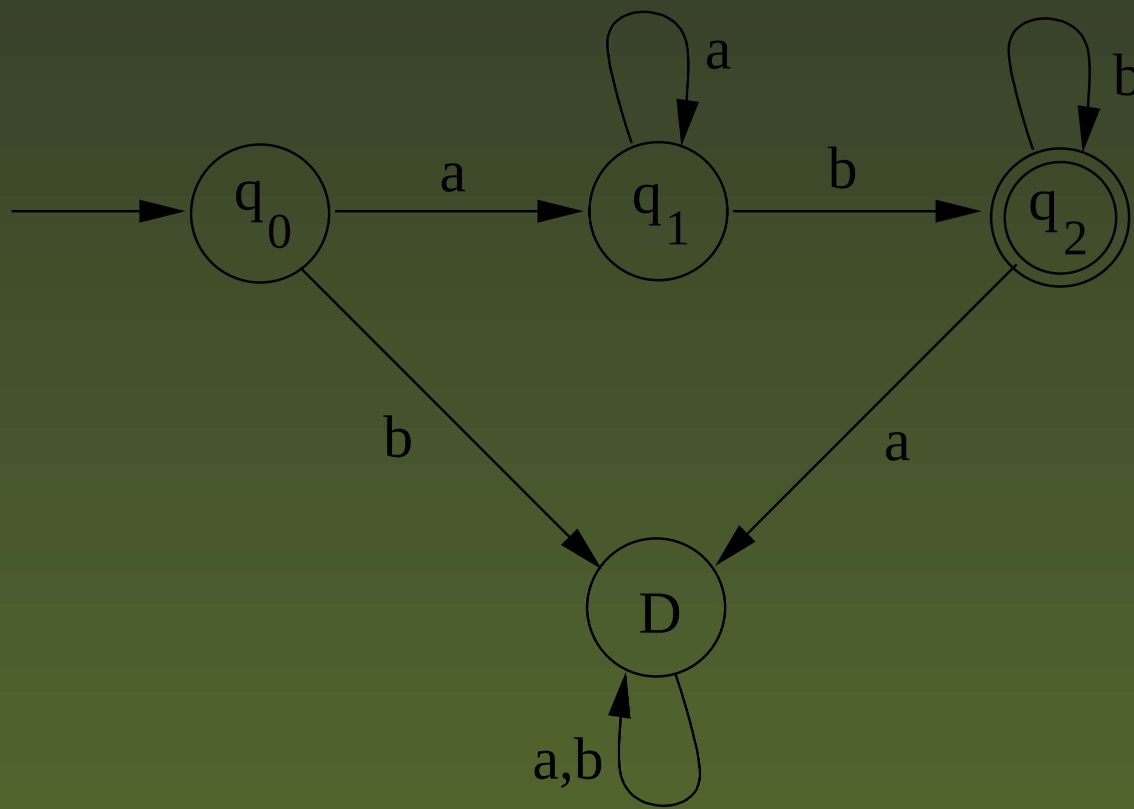
instant, moving the pointer one cell to the right.
$\hat{\delta}$ is an extension of δ and $\hat{\delta} : K \times \Sigma^* \rightarrow K$ as follows:

- $\hat{\delta}(q, \epsilon) = q$ for all q in K
- $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a) \qquad x \in \Sigma^*, q \in K, a \in \Sigma.$

Since $\hat{\delta}(q, a) = \delta(q, a)$, without any confusion we can use δ for $\hat{\delta}$ also. The language accepted by the automaton is defined as
$T(M) = \{w / w \in T^*, \delta(q_0, w) \in F\}.$

contd

Example Let a DFSA have state set $\{q_0, q_1, q_2, D\}$; q_0 is the initial state; q_2 is the only final state. The state diagram of the DFSA is given in the following figure.



contd

$$\begin{array}{l} a\ a\ a\ b \\ \uparrow \\ q_0 \end{array}$$

$$\begin{array}{llll} a & a & a & b \\ & \uparrow & & \\ & q_1 & & \end{array}$$

$$\begin{array}{llll} a & a & a & b \\ & & \uparrow & \\ & & q_1 & \end{array}$$

$$\begin{array}{lllll} a & a & a & b & \\ & & & \uparrow & \\ & & & q_1 & \end{array}$$

$$\begin{array}{lllll} a & a & a & b & \\ & & & & \uparrow \\ & & & & q_2 \end{array}$$

contd

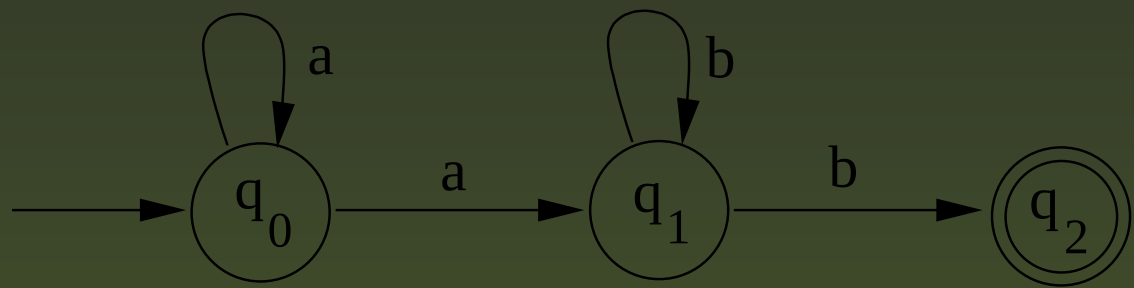
After reading $aaabb$, the automaton reaches a final state. It is easy to see that

$$T(M) = \{a^n b^m / n, m \geq 1\}$$

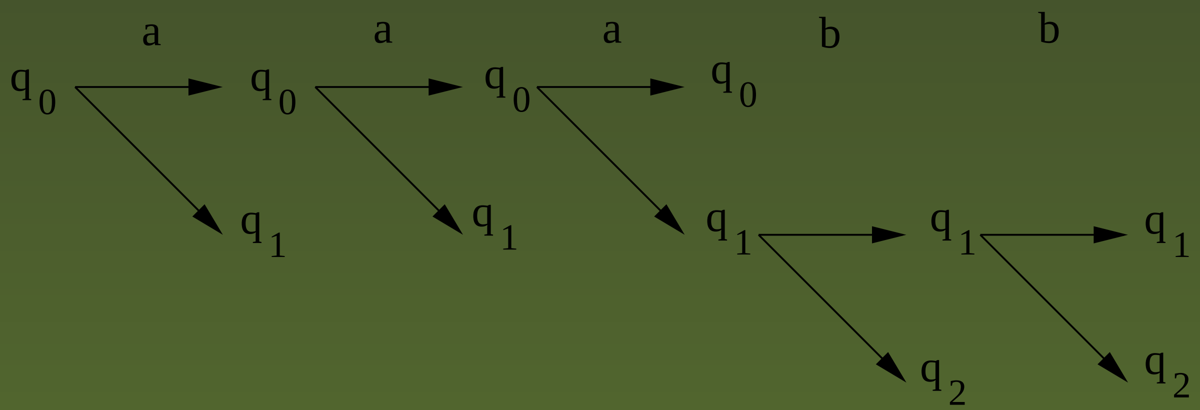
There is a reason for naming the fourth state as D. Once the control goes to D, it cannot accept the string, as from D the automaton cannot go to a final state. On further reading any symbol the state remains as D. Such a state is called a dead state or a sink state.

Nondeterministic Finite State Automaton

Consider the following state diagram of a nondeterministic FSA.



On a string *aaabb* the transition can be looked at as follows.



contd

Definition A Nondeterministic Finite State Automaton (NFSA) is a 5-tuple $M = (K, \Sigma, \delta, q_o, F)$ where $K, \Sigma, \delta, q_0, F$ are as given for DFSA and δ, the transition function is a mapping from $K \times \Sigma$ into finite subsets of K.
The mappings are of the form $\delta(q, a) = \{p_1, \ldots, p_r\}$ which means if the automaton is in state q and reads 'a' then it can go to any one of the states $p_1, \ldots, p_r$. δ is extended as $\hat{\delta}$ to $K \times \Sigma^*$ as follows:

$$\hat{\delta}(q, \epsilon) = \{q\} \text{ for all } q \text{ in } K.$$

contd

If P is a subset of K

$$\begin{aligned}
\delta(P, a) &= \bigcup_{p \in P} \delta(p, a) \\
\hat{\delta}(q, xa) &= \delta(\hat{\delta}(q, x), a) \\
\hat{\delta}(P, x) &= \bigcup_{p \in P} \hat{\delta}(p, x)
\end{aligned}$$

Since $\delta(q, a)$ and $\hat{\delta}(q, a)$ are equal for $a \in \Sigma$, we can use the same symbol δ for $\hat{\delta}$ also.

The set of strings accepted by the automaton is denoted by $T(M)$.

contd

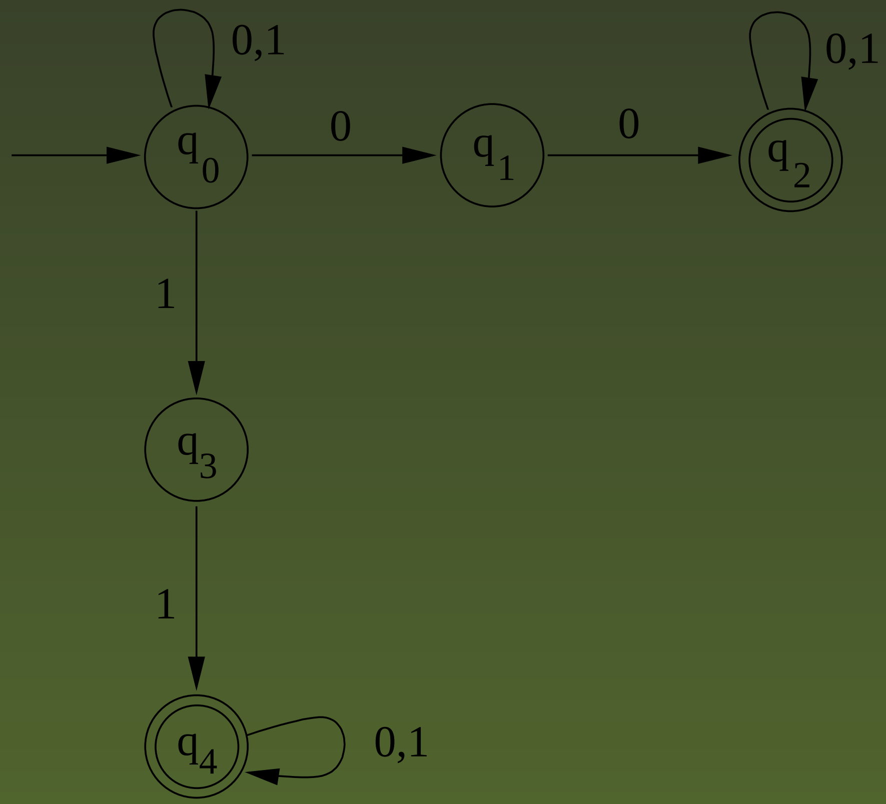
$T(M) = \{w/w \in T^*, \delta(q_0, w)$ contains a state from $F\}$

The automaton can be represented by a state table also. For example the state diagram given in the figure can be represented as the state table given below

	a	b
q_0	$\{q_0, q_1\}$	ϕ
q_1	ϕ	$\{q_0, q_1\}$
q_2	ϕ	ϕ

contd

Example The state diagram of an NFSA which accepts binary strings which have at least one pair '00' or one pair '11' is



contd

Theorem If L is accepted by a NFSA then L is accepted by a DFSA. Let L be accepted by a NFSA $M = (K, \Sigma, \delta, q_0, F)$. Then we construct a DFSA $M = (K', \Sigma', \delta', q_0', F')$ as follows: $K' = \mathbb{P}(K)$, power set of K. Corresponding to each subset of K, we have a state in K'. q_0' corresponds to the subset containing q_0 alone. F' consists of states corresponding to subsets having at least one state from F. We define δ' as follows:

$$\delta'([q_1, \ldots, q_k], a) = [r_1, r_2, \ldots, r_s] \text{ if and only if}$$

$$\delta(\{q_1, \ldots, q_k\}, a) = \{r_1, r_2, \ldots, r_s\}.$$

We show that $T(M) = T(M')$.

contd

We prove this by induction on the length of the string.
We show that
$\delta'(q_0', x) = [p_1, \ldots, p_r]$
if and only if $\quad \delta(q_0, x) = \{p_1, \ldots, p_r\}$

Basis

$|x| = 0$ i.e., $x = \epsilon$
$\delta'(q_0', \epsilon) = q_0' = [q_0]$
$\delta(q_0, \epsilon) = \{q_0\}$

Induction

Assume that the result is true for strings x of length upto m. We have to prove for string of length $m + 1$.
By induction hypothesis

contd

$\delta'(q_0', x) = [p_1, \ldots, p_r]$
if and only if $\delta(q_0, x) = \{p_1, \ldots, p_r\}$.
$\delta'(q_0', xa) = \delta'([p_1, \ldots, p_r], a)$,

$$\delta(q_0, xa) = \bigcup_{p \in P} \delta(p, a),$$

where $P = \{p_1, \ldots, p_r\}$.

Suppose $\displaystyle\bigcup_{p \in P} \delta(p, a) = \{s_1, \ldots, s_m\}$

$\delta(\{p_1, \ldots, p_r\}, a) = \{s_1, \ldots, s_m\}$.
By our construction
$\delta'([p_1, \ldots, p_r], a) = [s_1, \ldots, s_m]$ and hence
$\delta'(q_0', xa) = \delta'([p_1, \ldots, p_r], a) = [s_1, \ldots, s_m]$.

contd

In M', any state representing a subset having a state from F is in F'.

So if a string w is accepted in M, there is a sequence of states which takes M to a final state f and M' simulating M will be in a state representing a subset containing f. Thus $L(M) = L(M')$.

contd

Example Let us construct the DFSA for the NFSA given by the table in previous figure. We construct the table for DFSA.

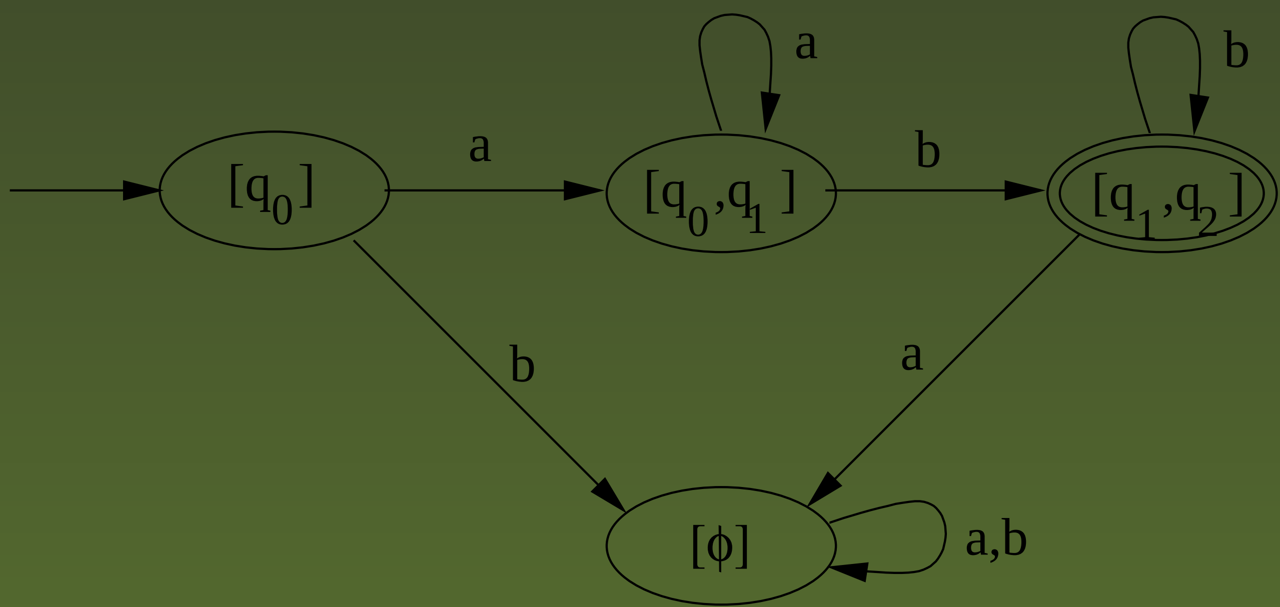
		a	b
$\rightarrow$	$[q_0]$	$[q_0, q_1]$	$[\phi]$
	$[q_0, q_1]$	$[q_0, q_1]$	$[q_1, q_2]$
	$[q_1, q_2]$	$[\phi]$	$[q_1, q_2]$
	$[\phi]$	$[\phi]$	$[\phi]$

$$\begin{aligned} \delta'([q_0, q_1], a) &= [\delta(q_0, a) \cup \delta(q_1, a)] \qquad (1) \\ &= [\{q_0, q_1\} \cup \phi] \qquad (2) \\ &= [q_0, q_1] \qquad (3) \end{aligned}$$

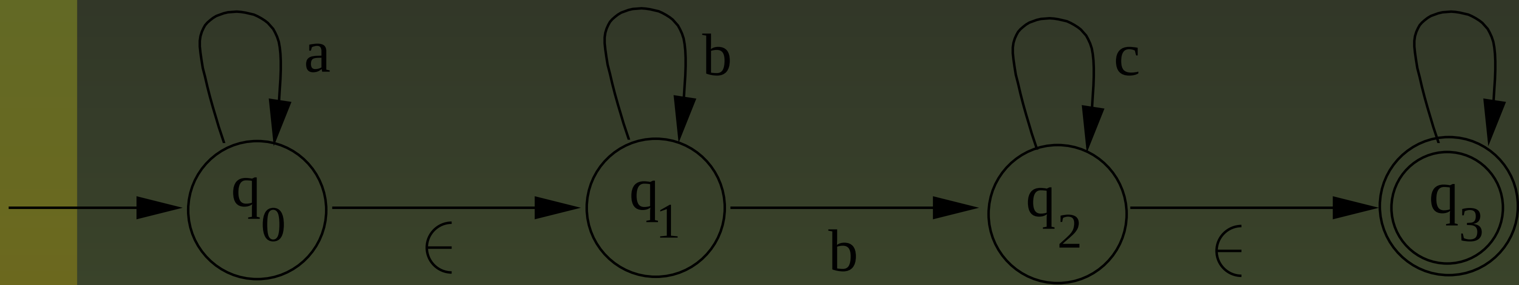
contd

$$\delta'([q_0, q_1], b) = [\delta(q_0, b) \cup \delta(q_1, b)] \tag{4}$$
$$= [\phi \cup \{q_1, q_2\}] \tag{5}$$
$$= [q_1, q_2] \tag{6}$$

The state diagram is given



Nondeterministic Finite State Automaton with ϵ-transitions



Definition An NFSA with ϵ-transition is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$. where $K, \Sigma, \delta, q_0, F$ are as defined for NFSA and δ is a mapping from $K \times (\Sigma \cup \{\epsilon\})$ into finite subsets of K. δ can be extended as $\hat{\delta}$ to $K \times \Sigma^*$ as follows. First we define the ϵ-closure of a state q. It is the set of states which can be reached from q by reading ϵ only. Of course, ϵ-closure of a state includes itself.
$\hat{\delta}(q, \epsilon) = \epsilon\text{-closure}(q)$.

contd

For w in Σ^* and a in Σ, $\hat{\hat{\delta}}(q, wa) = \epsilon$-closure$(P)$, where
$P = \{p|$ for some r in $\hat{\hat{\delta}}(q, w), p$ is in $\delta(r, a)\}$
Extending δ and $\hat{\hat{\delta}}$ to a set of states, we get
$\delta(Q, a) = \bigcup_{q\ in\ Q} \delta(q, a)$
$\delta(Q, w) = \bigcup_{q\ in\ Q} \delta(q, w)$
The language accepted is defined as

$$T(M) = \{w|\hat{\hat{\delta}}(q, w) \text{ contains a state in } F\}.$$

Theorem Let L be accepted by a NFSA with ϵ-moves. Then L can be accepted by a NFSA without ϵ-moves.
Let L be accepted by a NFSA with ϵ-moves
$M = (K, \Sigma, \delta, q_0, F)$. Then we construct a NFSA

contd

$M' = (K, \Sigma, \delta', q_0, F')$ without ϵ-moves for accepting L as follows.

$$\begin{aligned} F' &= F \cup \{q_0\} \text{ if } \epsilon\text{-closure of } q_0 \text{ contains a state from } F. \\ &= F \text{ otherwise.} \end{aligned}$$

$\delta'(q, a) = \hat{\delta}(q, a)$.
We should show $T(M) = T(M')$.

We wish to show by induction on the length of the string x accepted that $\delta'(q_0, x) = \hat{\delta}(q_0, x)$. We start the basis with $|x| = 1$ because for $|x| = 0$, i.e., $x = \epsilon$ this may not hold. We may have $\delta'(q_0, \epsilon) = \{q_0\}$ and $\hat{\delta}(q_0, \epsilon) = \epsilon$-closure of q_0 which may include other states.

contd

Basis

$|x| = 1$. Then x is a symbol of Σ say a, and

$\delta'(q_0, a) = \hat{\delta}(q_0, a)$ by our definition of δ'.

Induction

$|x| > 1$. Then $x = ya$ for some $y \in \Sigma^*$ and $a \in \Sigma$.

Then $\delta'(q_0, ya) = \delta'(\delta'(q_0, y), a)$.

By the inductive hypothesis $\delta'(q_0, y) = \hat{\delta}(q_0, y)$.

Let $\hat{\delta}(q_0, y) = P$.

contd

$$\delta'(P, a) = \bigcup_{p \in P} \delta'(p, a) = \bigcup_{p \in P} \hat{\delta}(p, a)$$

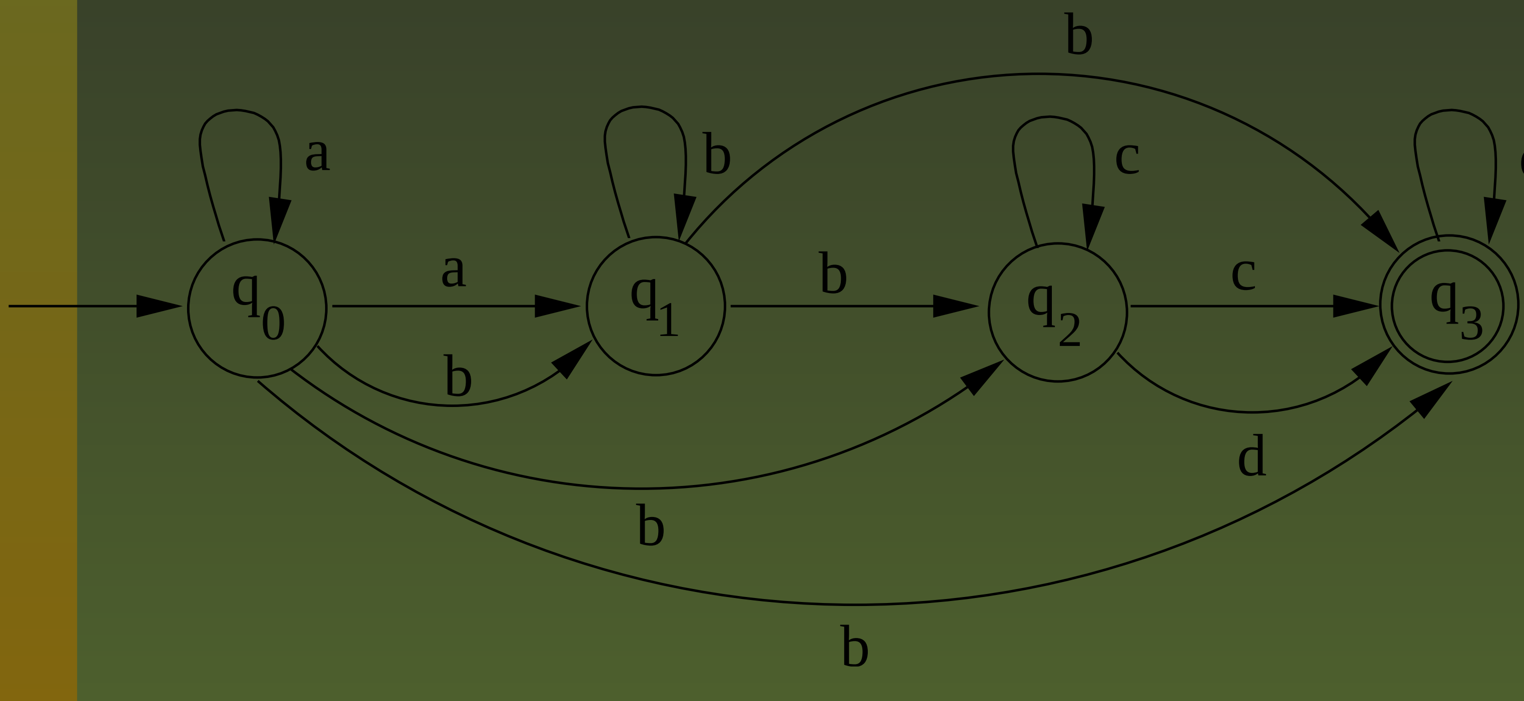
$$\bigcup_{p \in P} \hat{\delta}(p, a) = \hat{\delta}(q_0, ya)$$

$$\text{Therefore } \delta'(q_0, ya) = \hat{\delta}(q_0, ya)$$

It should be noted that $\delta'(q_0, x)$ contains a state in F' if and only if $\hat{\delta}(q_0, x)$ contains a state in F.

Example

Consider the ϵ-NFSA of previous example. By our construction we get the NFSA without ϵ-moves given in the following figure



contd

ϵ-closure of $(q_0) = \{q_0, q_1\}$
ϵ-closure of $(q_1) = \{q_1\}$
ϵ-closure of $(q_2) = \{q_2, q_3\}$
ϵ-closure of $(q_3) = \{q_3\}$
It is not difficult to see that the language accepted by the above $NFSA = \{a^n b^m c^p d^q / m \geq 1, n, p, q \geq 0\}$.

Regular Expressions

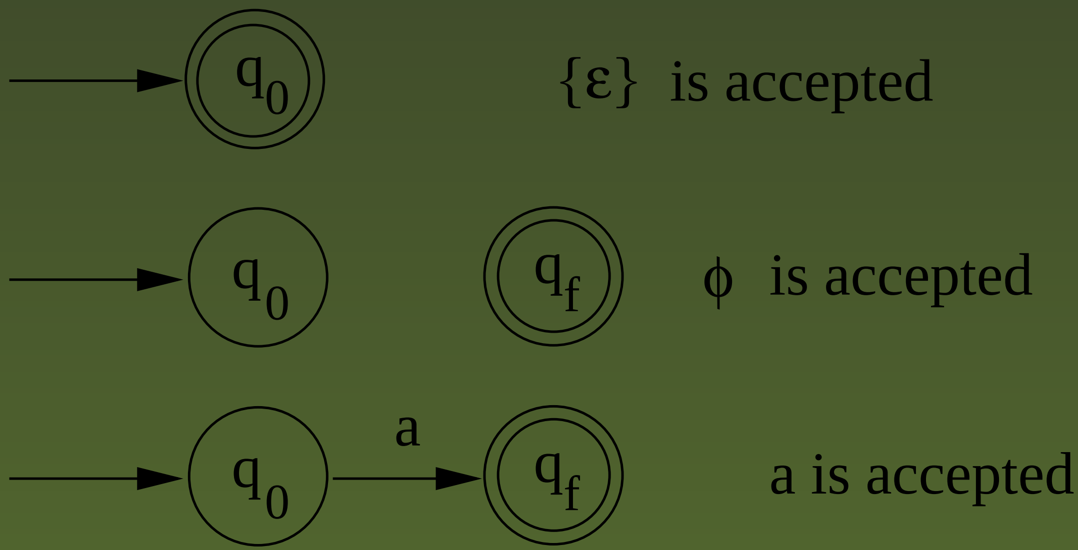
Definition Let Σ be an alphabet. For each a in Σ, **a** is a regular expression representing the regular set $\{a\}$. ϕ is a regular expression representing the empty set. ϵ is a regular expression representing the set $\{\epsilon\}$. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are regular expressions representing the regular sets R_1 and R_2 respectively, then $\mathbf{r}_1 + \mathbf{r}_2$ is a regular expression representing $R_1 \cup R_2$. $\mathbf{r}_1\mathbf{r}_2$ is a regular expression representing $R_1 R_2$. $\mathbf{r}_1^*$ is a regular expression representing R_1^*. Any expression obtained from ϕ, ϵ, $a(a \in \Sigma)$ using the above operations and parentheses where required is a regular expression.

Example $(ab)^*abcd$ represent the regular set

$$\{(ab)^n cd / n \geq 1\}$$

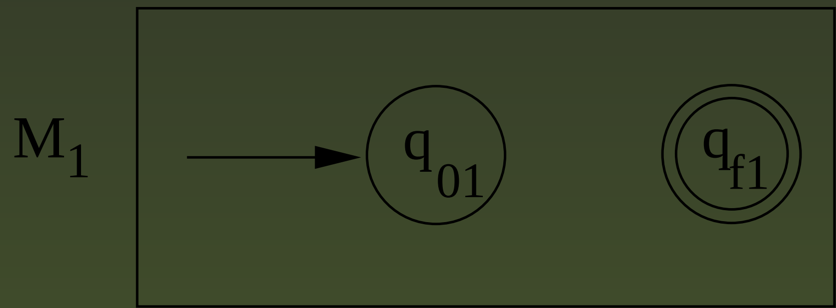
contd

Theorem If **r** is a regular expression representing a regular set, we can construct an NFSA with ϵ-moves to accept **r**.
r is obtained from **a**, $(a \in \Sigma)$, ϵ, ϕ by finite number of applications of $+$, . and $*$ (. is usually left out).
For ϵ, ϕ, **a** we can construct NFSA with ϵ-moves are.

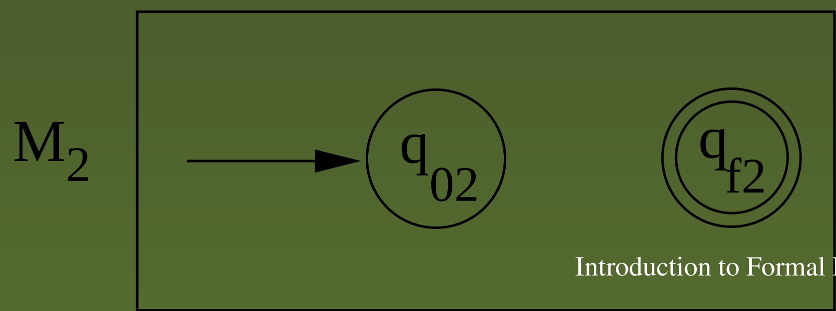


contd

Let $\mathbf{r}_1$ represent the regular set R_1 and R_1 is accepted by the NFSA M_1 with ϵ-transitions.

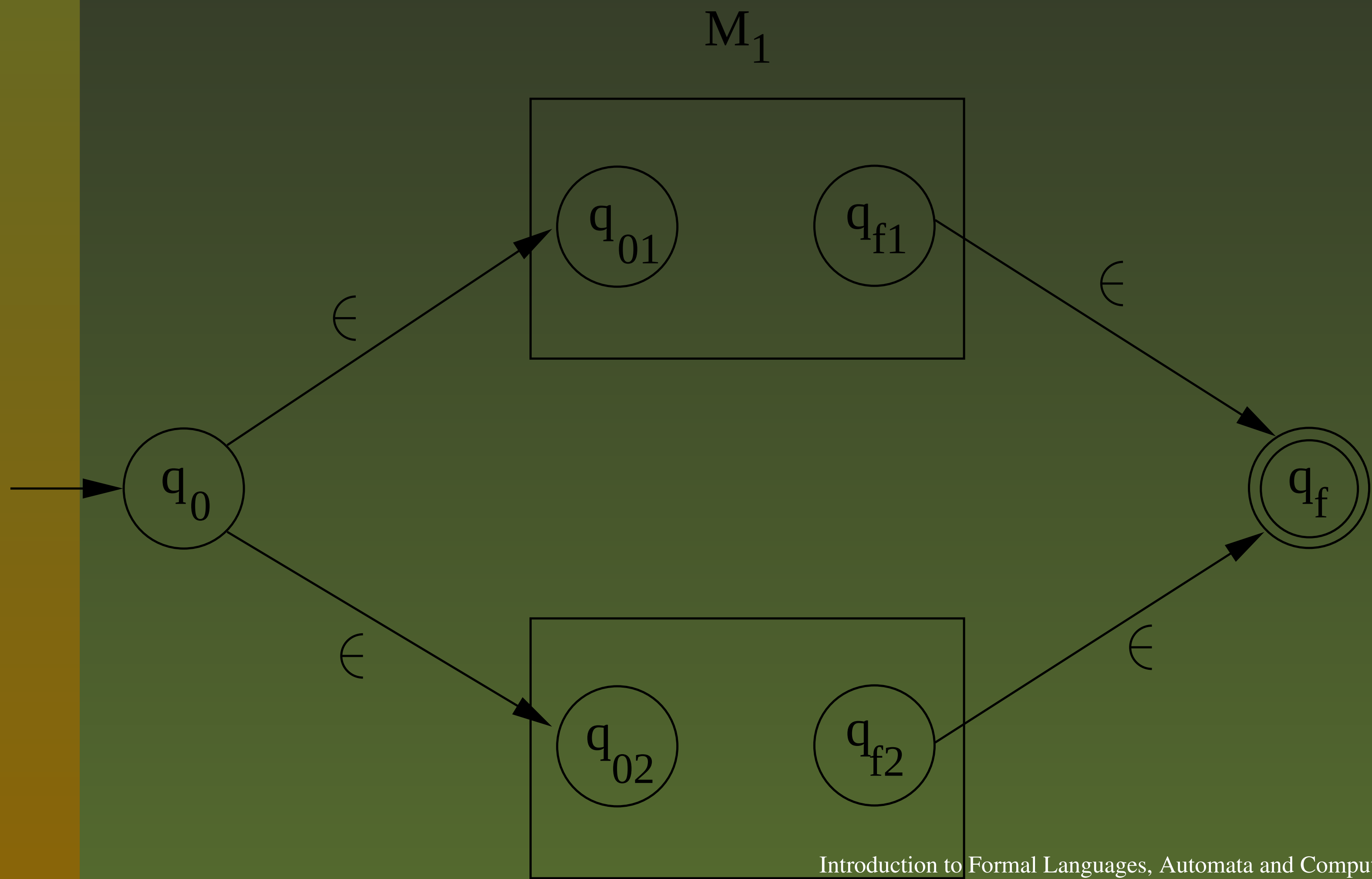


Without loss of generality we can assume that each such NFSA with ϵ-moves has only one final state.
R_2 is similarly accepted by an NFSA M_2 with ϵ-transition.



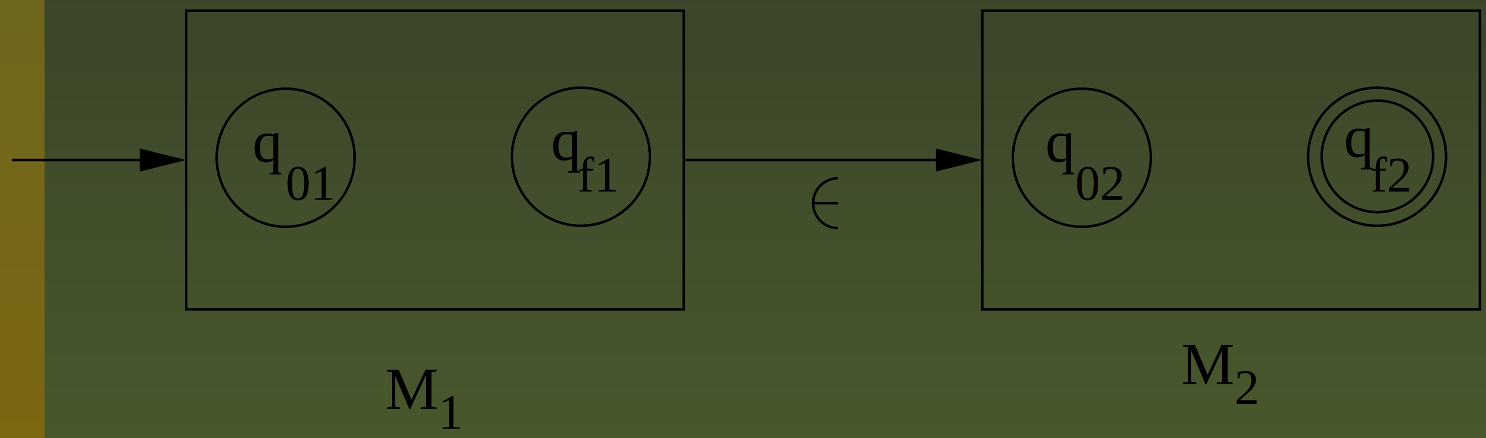
contd

Now we can easily see that $R_1 \cup R_2$ (represented by $\mathbf{r}_1 + \mathbf{r}_2$) is accepted by the NFSA given in next figure



contd

For this NFSA q_0 is the start state and q_f is the final state.
$R_1 R_2$ represented by $\mathbf{r}_1\mathbf{r}_2$ is accepted by the NFSA with ϵ-moves given as



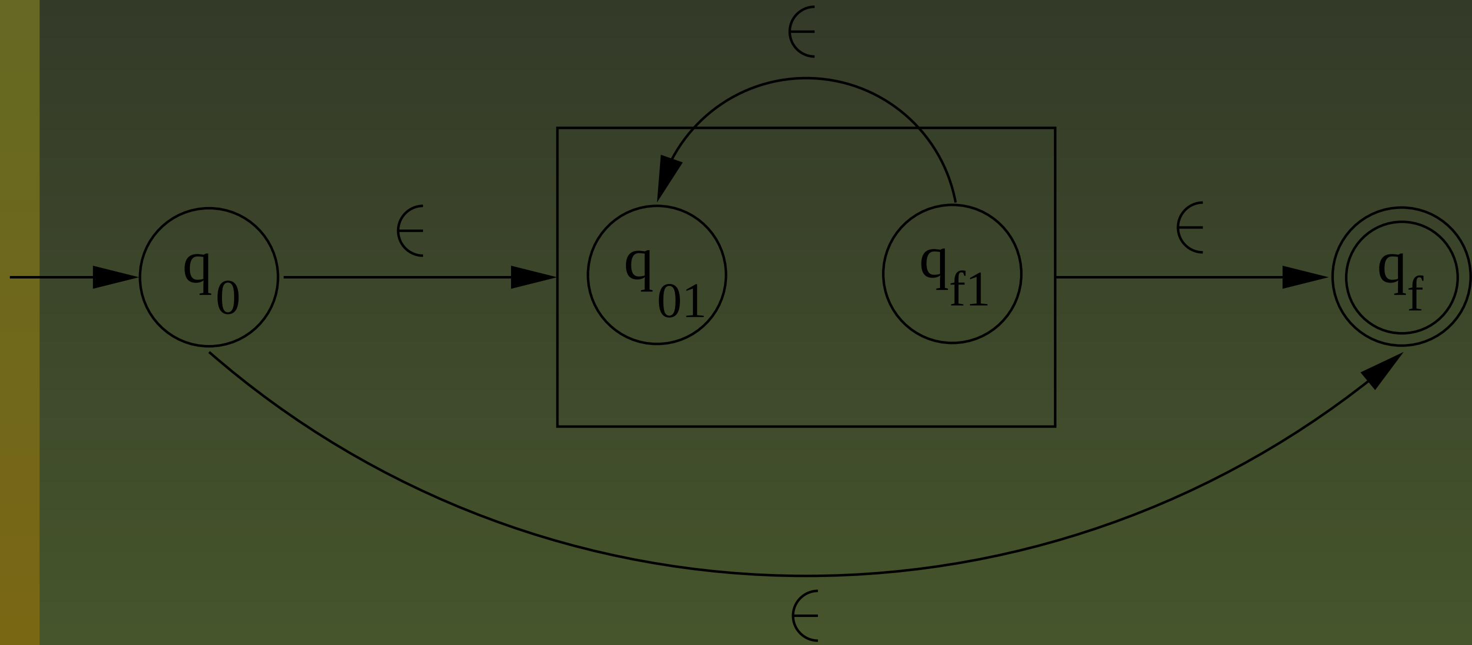
For this NFSA with ϵ-moves q_{01} is the initial state and $q_{f2}$is the final state.

$R_1^* = R_1^0 \cup R_1^1 \cup R_1^2 \cup \cdots \cup R_1^k \cup \ldots$

$R_1^0 = \{\epsilon\}$ and $R_1' = R_1$.

contd

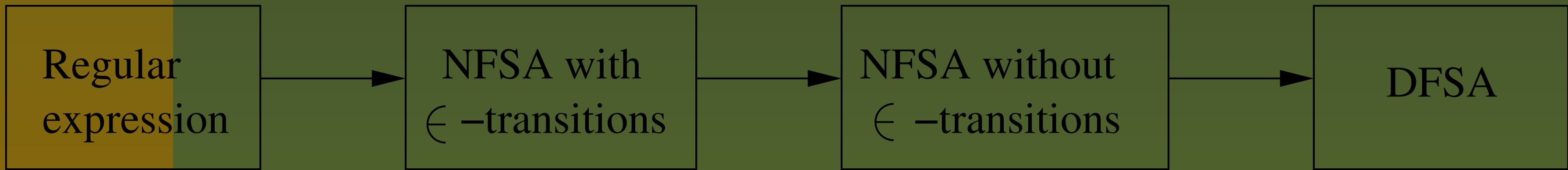
R_1^* represented by $\mathbf{r}_1^*$ is accepted by the NFSA with ϵ-moves given as



For this NFSA with ϵ-moves q_0 is the initial state and q_f is the final state. It can be seen that R_1^* contains strings of the form $x_1, x_2, \ldots, x_k$ each $x_i \in R_1$. To accept

contd

this string, the control goes from q_0 to q_{01} and then after reading x_1 and reaching q_{f1}, it goes to q_{01}, by an ϵ-transition. From q_{01}, it again reads x_2 and goes to q_{f1}. This can be repeated a number (k) of times and finally the control goes to q_f from q_{f1} by an ϵ-transition. $R_1^0 = \{\epsilon\}$ is accepted by going to q_f from q_0 by an ϵ-transition.
Thus we have seen that given a regular expression one can construct an equivalent NFSA with ϵ-transitions.

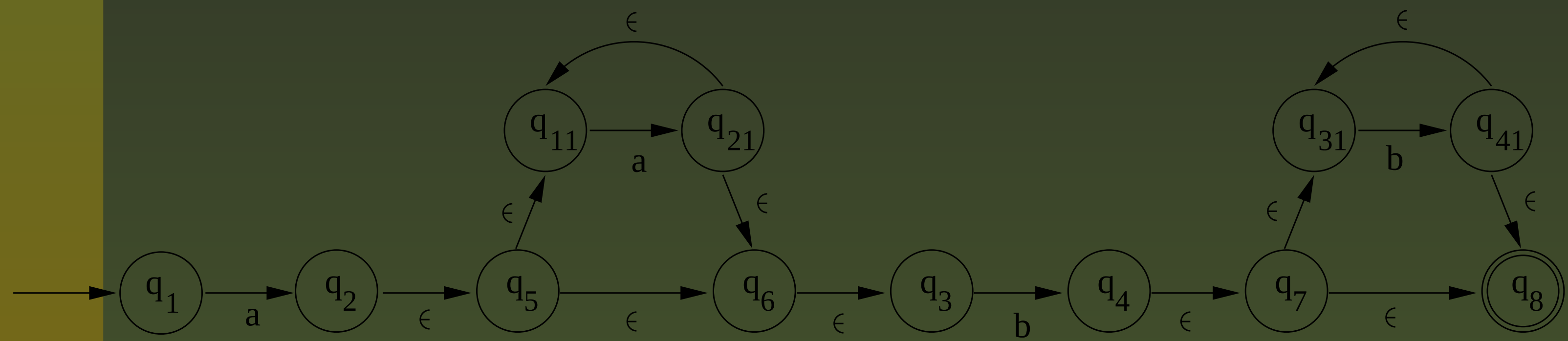


Example

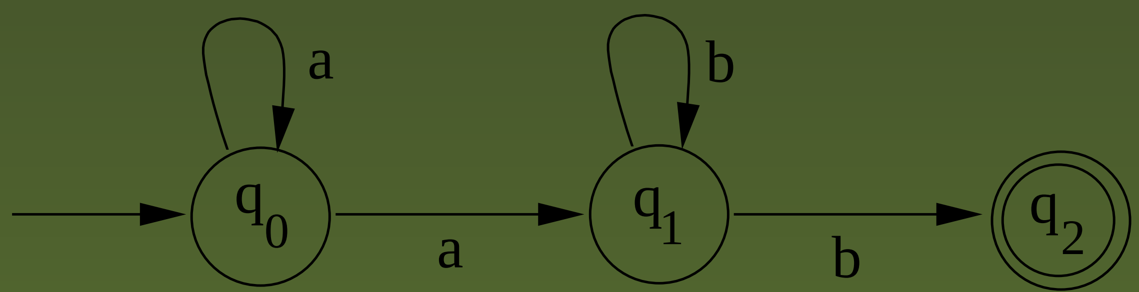
Consider a regular expression aa^*bb^*.
a and b are accepted by NFSA with ϵ-moves given in the following figure

contd

aa^*bb^* will be accepted by NFSA with ϵ-moves given in next figure.



But we have already seen that a simple NFSA can be drawn easily for this as in next figure.



contd

Definition Let $L \subseteq \Sigma^*$ be a language and x be a string in Σ^*. Then the derivative of L with respect to x is defined as

$$L_x = \{y \in \Sigma^* / xy \in L\}.$$

It is some times denoted as $\partial_x L$.
Theorem If L is a regular set L_x is regular for any x.
Consider a DFSA accepting L. Let this FSA be $M = (K, \Sigma, \delta, q_0, F)$. Start from q_0 and read x to go to state $q_x \in K$.

Then $M' = (K, \Sigma, \delta, q_x, F)$ accepts L_x . This can be seen easily as below.

contd

$\delta(q_0, x) = q_x$,
$\delta(q_0, xy) \in F \Leftrightarrow xy \in L$,
$\delta(q_0, xy) = \delta(q_x, y)$,
$\delta(q_x, y) \in F \Leftrightarrow y \in L_x$,
$\therefore M'$ accepts L_x.

lemma Let Σ be an alphabet. The equation $X = AX \cup B$ where $A, B \subseteq \Sigma^*$ has a unique solution A^*B if $\epsilon \notin A$.

contd

Let

$$
\begin{aligned}
X &= AX \cup B \\
&= A(AX \cup B) \cup B \\
&= A^2X \cup AB \cup B \\
&= A^2(AX \cup B) \cup AB \cup B \\
&= A^3X \cup A^2B \cup AB \cup B \\
&\vdots \\
&= A^{n+1}X \cup A^nB \cup A^{n-1}B \cup \cdots \cup AB \cup B \qquad (7)
\end{aligned}
$$

Since $\epsilon \notin A$, any string in A^k will have minimum length k.

To show $X = A^*B$.

contd

Let $w \in X$ and $|w| = n$. We have

$$X = A^{n+1}X \cup A^n B \cup \cdots \cup AB \cup B \tag{8}$$

Since any string in $A^{n+1}X$ will have minimum length $n+1$, w will belong to one of $A^k B$, $k \leq n$. Hence $w \in A^*B$. On the other hand let $w \in A^*B$. To prove $w \in X$. Since $|w| = n$, $w \in A^k B$ for some $k \leq n$. Therefore from (8) $w \in X$.
Hence we find that the unique solution for $X = AX + B$ is $X = A^*B$.

Note If $\epsilon \in A$, the solution will not be unique. Any A^*C, where $C \supseteq B$, will be a solution.

contd

Next we give an algorithm to find the regular expression corresponding to a DFSA.

Algorithm

Let $M = (K, \Sigma, \delta, q_0, F)$ be the DFSA.

$\Sigma = \{a_1, a_2, \ldots, a_k\}, \qquad K = \{q_0, q_1, \ldots, q_{n-1}\}.$

Step 1 Write an equation for each state in K.

$q = a_1 q_{i1} + a_2 q_{i2} + \cdots + a_k q_{ik}$

if q is not a final state and $\delta(q, a_j) = q_{ij} \quad 1 \leq j \leq k.$

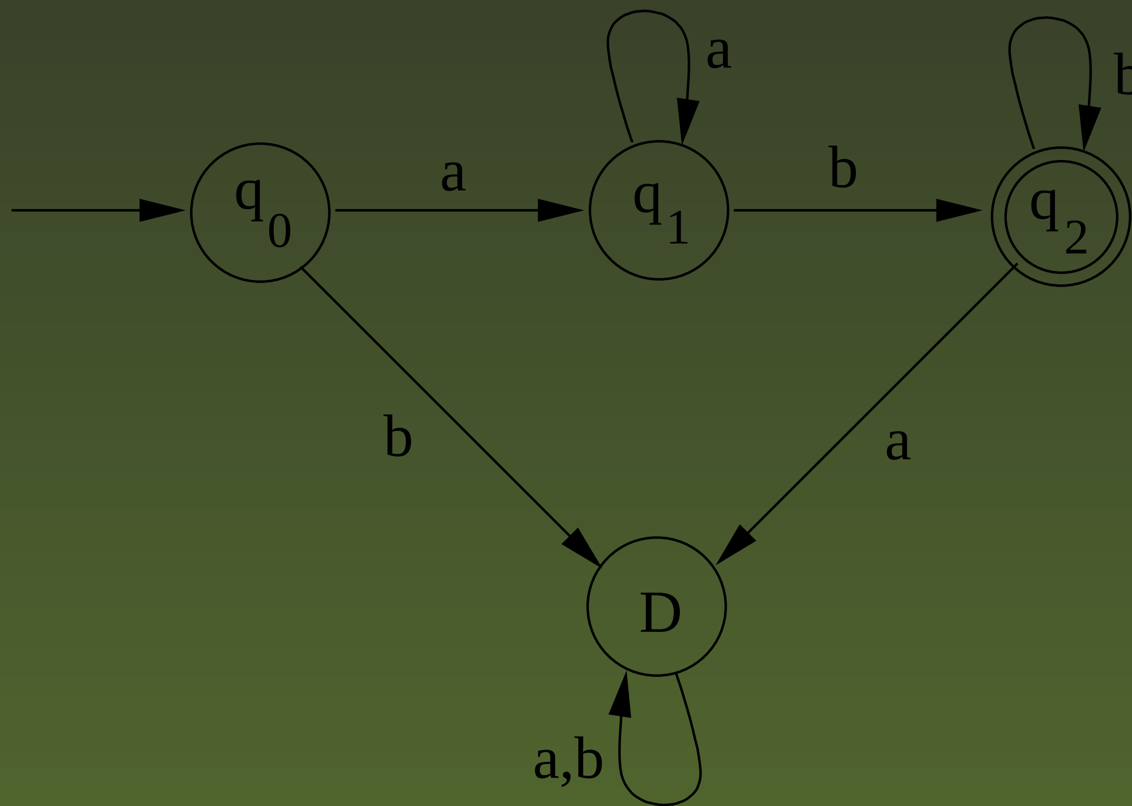
$q = a_1 q_{i1} + a_2 q_{i2} + \cdots + a_k q_{ik} + \lambda$

if q is a final state and $\delta(q, a_j) = q_{ij} \quad 1 \leq j \leq k.$

Step 2 Take the n equations with n variables q_i, $1 \leq i \leq n$, and solve for q_0 using the above lemma and substitution.

contd

Step 3 Solution for q_0 gives the desired regular expression. Let us execute this algorithm for the following DFSA given in the figure.



contd

Step 1

$$q_0 = aq_1 + bD \tag{9}$$

$$q_1 = aq_1 + bq_2 \tag{10}$$

$$q_2 = aD + bq_2 + \lambda \tag{11}$$

$$D = aD + bD \tag{12}$$

Step 2
Solve for q_0. From (12)

$$D = (a + b)D + \phi$$

contd

Using previouus lemma

$$D = (a+b)^* \phi = \phi. \tag{13}$$

Using them we get

$$q_0 = aq_1 \tag{14}$$

$$q_1 = aq_1 + bq_2 \tag{15}$$

$$q_2 = bq_2 + \lambda \tag{16}$$

Note that we have got rid of one equation and one variable.

contd

In 16 using the lemma we get

$$q_2 = b^* \tag{17}$$

Now using 17 and 15

$$q_1 = aq_1 + bb^* \tag{18}$$

We now have 14 and 18. Again we eliminated one equation and one variable.
Using the above lemma in (18)

$$q_1 = a^*bb^* \tag{19}$$

contd

Using 19 in 14

$$q_0 = aa^*bb^* \tag{20}$$

This is the regular expression corresponding to the given FSA.
Next, we see, how we are justified in writing the equations.
Let q be the state of the DFSA for which we are writing the equation,

$$q = a_1 q_{i1} + a_2 q_{i2} + \cdots + a_k q_{ik} + Y. \tag{21}$$

$$Y = \lambda \text{ or } \phi.$$

contd

Let L be the regular set accepted by the given DFSA. Let x be a string such that starting from q_0, after reading x, state q is reached. Therefore q represents L_x, the derivative of L with respect to x. From q after reading a_j, the state q_{ij} is reached.

$$L_x = q = a_1 L_{xa_1} + a_2 L_{xa_2} + \cdots + a_k L_{xa_k} + Y. \quad (22)$$

$a_j L_{xa_j}$ represents the set of strings in L_x beginning with a_j. Hence equation (22) represents the partition of L_x into strings beginning with a_1, beginning with a_2 and so on. If L_x contains ϵ, then $Y = \epsilon$ otherwise $Y = \phi$.

contd

It should be noted that when L_x contains ϵ, q is a final state and so $x \in L$. It should also be noted that considering each state as a variable q_j, we have n equation in n variables. Using the above lemma, and substitution, each time one equation is removed while one variable is eliminated. The solution for q_0 is $L_\epsilon = L$. This gives the required regular expression.