Finite element method for structural dynamic and stability analyses

Module-11

Nonlinear FE Models

Lecture-37 Introduction and review of continuum mechanics

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Linear systems & principle of superposition

Additivity property

Scaling property

for all complex $a$
Nonlinear system: a system that is not linear

- A nonspecific description

\[
y = mx + c \\
y_1 = max_1 + c \\
ay = amx_1 + ac \neq y_1
\]

System is not linear since it does not obey scaling property.
Zero input produces zero output in a linear system.
This is not satisfied in this case.
\[ \ddot{y} + 2\eta \omega \dot{y} + \omega^2 y = x(t); \quad y(0) = 0, \dot{y}(0) = 0 \]
\[ \ddot{y}_1 + 2\eta \omega \dot{y}_1 + \omega^2 y_1 = x_1(t); \quad y_1(0) = 0, \dot{y}_1(0) = 0 \]
\[ \ddot{y}_2 + 2\eta \omega \dot{y}_2 + \omega^2 y_2 = x_2(t); \quad y_2(0) = 0, \dot{y}_2(0) = 0 \]
\[ \Rightarrow (\ddot{y}_1 + \ddot{y}_2) + 2\eta \omega (\dot{y}_1 + \dot{y}_2) + \omega^2 (y_1 + y_2) = x_1(t) + x_2(t) \]
\[ \Rightarrow (a\ddot{y}_1) + 2\eta \omega (a\dot{y}_1) + \omega^2 (ay_1) = \left[ ax_1(t) \right]; ay_1(0) = 0, a\dot{y}_1(0) = 0 \]
\[ \Rightarrow \text{system is linear} \]

\[ \ddot{y} + 2\eta \omega \dot{y} + \omega^2 y + \alpha y^3 = x(t); \quad y(0) = 0, \dot{y}(0) = 0 \]
\[ \ddot{y}_1 + 2\eta \omega \dot{y}_1 + \omega^2 y_1 + \alpha y_1^3 = x_1(t); \quad y_1(0) = 0, \dot{y}_1(0) = 0 \]
\[ \ddot{y}_2 + 2\eta \omega \dot{y}_2 + \omega^2 y_2 + \alpha y_2^3 = x_2(t); \quad y_2(0) = 0, \dot{y}_2(0) = 0 \]
\[ \Rightarrow (\ddot{y}_1 + \ddot{y}_2) + 2\eta \omega (\dot{y}_1 + \dot{y}_2) + \omega^2 (y_1 + y_2) + \alpha (y_1^3 + y_2^3) = x_1(t) + x_2(t) \]
\[ \ddot{y}_3 + 2\eta \omega \dot{y}_3 + \omega^2 y_3 + \alpha y_3^3 = x_1(t) + x_2(t); \quad y_3(0) = 0, \dot{y}_3(0) = 0 \]

Clearly, \( y_3 \neq (y_1 + y_2) \Rightarrow \text{system is nonlinear} \)
Exercise
Examine the previous two examples by including non-zero initial conditions.

\[ y(t) = \frac{1}{x(t)} \left( \frac{dx}{dt} \right)^2 \]

\[ x(t) \to ax(t) \Rightarrow y_1(t) = \frac{a^2}{ax(t)} \left( \frac{dx}{dt} \right)^2 = ay(t) \]

Scaling property is satisfied.
Additivity property is not satisfied (verify).
System is not linear.
Qualitative features of nonlinear dynamic response

• Response to all the loads needs to be analyzed simultaneously
• Undamped free vibration: frequency of oscillations depends upon initial conditions.
• Harmonic inputs at $\Omega$ can produce
  - harmonic response at frequencies $\neq \Omega$
  - non-harmonic responses
  - aperiodic responses
  \[ \text{primary, subharmonic and super harmonic resonances} \]
• Reciprocity relations are not valid
• Large responses can occur at frequencies other than the driving frequencies
• Steady state responses depend upon initial conditions
• Multiplicity of steady state solutions are possible
• System can possess multiple equilibrium states and display a wide range of bifurcations
• Concept of normal modes, natural frequencies, and natural coordinates no longer applicable
• Band limited excitations can produce responses with frequency content outside the bandwidth of the excitation.

...
Example

\[ \ddot{x} + \omega_n^2 x + \mu x^3 = 0; x(0) = A \& \dot{x}(0) = 0 \]

\[ \mu = 0 \Rightarrow x_0(t) = A \cos \omega_n t \]

\[ \mu \neq 0 \Rightarrow \text{The frequency of oscillation and the nature of oscillations change.} \]

\[ x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \]

\[ \omega^2 = \omega_n^2 + \mu \alpha_1 + \mu^2 \alpha_2 + \cdots \]

Unknowns: \(\{x_1(t), x_2(t), \cdots\}\) \& \(\{\alpha_1, \alpha_2, \cdots\}\)

\[ \{\ddot{x}_0(t) + \mu \ddot{x}_1(t) + \mu^2 \ddot{x}_2(t) + \cdots\} + \]

\[ \left[ \omega^2 - \mu \alpha_1 - \mu^2 \alpha_2 + \cdots \right] \{x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots\} + \]

\[ \mu \left\{ x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \right\}^3 = 0 \]

\[ \mu^0 : \ddot{x}_0(t) + \omega^2 x_0(t) = 0 \]

\[ \mu^1 : \dddot{x}_1(t) + \omega^2 x_1(t) - \alpha_1 x_0(t) - x_0^3(t) = 0 \]
\[ \ddot{x}_0 (t) + \omega^2 x_0 (t) = 0 \Rightarrow x_0 (t) = A \cos \omega t \]
\[ \ddot{x}_1 (t) + \omega^2 x_1 (t) = \alpha_1 A \cos \omega t + A^3 \cos^3 \omega t \]
\[ = \left( \alpha_1 - \frac{3}{4} A^2 \right) A \cos \omega t - \frac{A^3}{4} \cos 3\omega t \]

If \( \left( \alpha_1 - \frac{3}{4} A^2 \right) \neq 0 \), \( \lim_{t \to \infty} x_1 (t) \to \infty \)

This is physically not valid \( \Rightarrow \left( \alpha_1 - \frac{3}{4} A^2 \right) = 0 \Rightarrow \alpha_1 = \frac{3}{4} A^2 \)

\[ x_1 (t) = C_1 \sin \omega t + C_2 \cos \omega t + \frac{A^3}{32 \omega^2} \cos 3\omega t \]
\[ x_1 (0) = 0 \& \dot{x}_1 (0) = 0 \Rightarrow x_1 (t) = \frac{A^3}{32 \omega^2} (\cos 3\omega t - \cos \omega t) \]
\[ x(t) = A \cos \omega t + \frac{\mu A^3}{32\omega^2} (\cos 3\omega t - \cos \omega t) \]

\[ \omega = \omega_n \sqrt{1 + \frac{3\mu}{4} A^2} \]

Remark
- Solution is periodic (but not harmonic)
- Period depends upon initial condition
- \( \mu > 0 \Rightarrow \omega > \omega_0 \) (hardening)
- \( \mu < 0 \Rightarrow \omega < \omega_0 \) (softening)
- Solution can be improved by including higher order terms
Example

\[ \ddot{x} + c\dot{x} + \omega_n^2 x + \mu x^3 = F \cos(\omega t + \phi); x(0) = x_0 \& \dot{x}(0) = \dot{x}_0 \]

Consider steady state response of the form

\[ x_0(t) = A \cos \omega t \]

\[ \Rightarrow -A\omega^2 \cos \omega t - cA\omega \sin \omega t + \omega_n^2 A \cos \omega t + \mu A^3 \cos^3 \omega t \]

\[ = F \left( \cos \omega t \cos \phi - \sin \omega t \sin \phi \right) \]

\[ \Rightarrow \left[ (\omega_n^2 - \omega^2) A + \frac{3}{4} \mu A^3 \right] \cos \omega t - c\omega A \sin \omega t + \frac{\mu A^3}{4} \cos 3\omega t \]

\[ = A_0 \cos \omega t - B_0 \sin \omega t \]

\[ \Rightarrow \left[ (\omega_n^2 - \omega^2) A + \frac{3}{4} \mu A^3 \right] = A_0 \& c\omega A = B_0 \]

\[ \Rightarrow F^2 = \left[ (\omega_n^2 - \omega^2) + \frac{3}{4} \mu A^3 \right]^2 + [c\omega A]^2 \]
Stability of solutions

\[ u(t) = x_0(t) + \xi(t) \]

\[ \Rightarrow \dot{x}_0(t) + \ddot{\xi}(t) + c\{\dot{x}_0(t) + \dot{\xi}(t)\} + \omega_n^2 \{x_0(t) + \xi(t)\} + \mu\{x_0(t) + \xi(t)\}^3 \]

\[ = F \cos(\omega t + \phi) \]

\[ \Rightarrow \ddot{\xi} + c\dot{\xi} + \xi(t)\{\omega_n^2 + 3\mu A^2 \cos^2 \omega t\} = 0 \]

Time varying system with periodic coefficient
Use Floquet's theory to infer stability of the solutions.

Steady state solutions can be multivalued.
The steady states are functions of initial conditions.

\[ \ddot{x} + 2\varepsilon\mu\dot{x} + \omega_0^2 x + \varepsilon\alpha x^3 = F \cos \Omega t; \quad x(0) = x_0 \& \dot{x}(0) = \dot{x}_0 \]

\[ \Omega \approx \omega_0 \]

\[ \Omega = \omega_0 + \varepsilon\sigma \]

Detuning

\[ \ddot{x} + 2\varepsilon\mu\dot{x} + \omega_0^2 x + \varepsilon\alpha x^3 = \varepsilon k \cos \Omega t \]

Method of multiple scales ⇒

\[ x(t) = a(t) \cos \left[ \omega_0 t + \beta(t) \right] \]

\[ a' = -\mu a + \frac{1}{2} \frac{k}{\omega_0} \sin (\sigma \varepsilon t - \beta) \]

\[ a\beta' = \frac{3}{8} \frac{\alpha}{\omega_0} a^3 - \frac{1}{2} \frac{k}{\omega_0} \cos (\sigma \varepsilon t - \beta) \]

Let \( \gamma = (\sigma \varepsilon t - \beta) \)
\[
\begin{align*}
& \lim_{t \to \infty} a' \to 0 \quad \text{and} \quad \lim_{t \to \infty} \beta' \to 0 \implies \\
& -\mu a + \frac{1}{2 \omega_0} k \sin \gamma = 0 \\
& \frac{3}{8 \omega_0} a^3 - \frac{1}{2 \omega_0} k \cos \gamma = 0 \\
& \implies \left[ \mu^2 + \left( \sigma - \frac{3 \alpha}{8 \omega_0} a^2 \right)^2 \right] a^2 = \frac{k^2}{4 \omega_0^2}
\end{align*}
\]

Multi-valued frequency response function

Stability

Investigate the fixed points and stability of the following system of equations:

\[
\begin{align*}
a' &= -\mu a + \frac{1}{2 \omega_0} k \sin(\sigma \varepsilon t - \beta) \\
\alpha \beta' &= \frac{3}{8 \omega_0} a^3 - \frac{1}{2 \omega_0} k \cos(\sigma \varepsilon t - \beta)
\end{align*}
\]
Importance of nonlinear analysis

- **Earthquake engineering:**
  - Structures are to be designed to display controlled inelastic responses; there are certain preferred modes of failures and certain failure modes are not preferred.
  - Use of snubbers, restraint devices, isolators, and nonlinear energy dissipation devices (e.g., slotted bolt connections)

- **Wind engineering:** wind structure interactions; across wind oscillations; galloping.

- **Materials** like concrete and soil display nonlinear behaviour even at low values of strains. Differing behaviour in tension and compression. Response depends upon entire time history, duration over which the load is applied and ambient effects such as temperature.

- **Vibration** of cracked structures.

- **Study of failures.**

- **Loss of stability**

- **Prototype testing** using nonlinear FE models (e.g., crash analysis in automotive design, simulation of drop test in electronics industry)
Sources of nonlinearity

• Nonlinear strain-displacement relations (geometric nonlinearity)
• Nonlinear constitutive laws (nonlinear stress-strain relations)
• Nonlinearity associated with boundary conditions
• Nonlinear energy dissipation mechanisms
Material nonlinear; small displacements and small strains

Material linear/nonlinear; large rotation and small strains
Material linear/nonlinear; large rotation and large strains

Nonlinear boundary conditions

Stress vs. Strain graph
Nonlinearly elastic systems and systems with hereditary nonlinearities

\[ m\ddot{x} + c\dot{x} + kx + g[x(t), \dot{x}(t), t] + h[x(\tau), \dot{x}(\tau); 0 \leq \tau \leq t] = f(t); \]
\[ x(0) \& \dot{x}(0) \text{ specified} \]
\[ g[x(t), \dot{x}(t), t] = \text{Nonlinear function of instantaneous values of } x(t) \& \dot{x}(t). \]
\[ h[x(\tau), \dot{x}(\tau); 0 \leq \tau \leq t] = \text{Nonlinear function of response time histories up to time } t. \]
Nonlinear effects in wind induced oscillations
e.g., Across wind oscillations of chimneys

\[ m\ddot{x} + c\dot{x} + kx - k_a \dot{x}(1 - \varepsilon \dot{x}^2) = F(t) \]

- Limit cycles
- Entrainment
Objectives

• A brief review of background concepts

• Present a flavour of treatment of nonlinear structural mechanics problems using finite element method.

• Focus on geometrically nonlinear problems
Planar beam element

- Large transverse displacements
- Small strains
- Moderate rotations

Changes in geometry due to deformation is not accounted for in defining stress.

Euler-Bernoulli beam: Upon deformation \( mn \) remains straight and normal to the neutral axis and its length does not change.

\[
\begin{align*}
    u_1 &= u_0(x) - z \frac{dw_0}{dx} \\
    u_2 &= 0 \\
    u_3 &= w_0(x)
\end{align*}
\]

\( u_1, u_2, u_3 = \) displacement along \( x, y, z \) respectively

\( u_0(x) = \) axial displacement of a point on neutral axis

\( w_0(x) = \) axial displacement of a point on neutral axis

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j}
\]
\[ u_1 = u_0(x) - z \frac{dw_0}{dx}; u_2 = 0; u_3 = w_0(x) \]

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right] = \left( \frac{du_0}{dx} - z \frac{d^2 w_0}{dx^2} \right) + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \]

\[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 + \left( \frac{\partial u_3}{\partial x_2} \right)^2 \right] = 0 \]

\[ \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_3} \right)^2 + \left( \frac{\partial u_2}{\partial x_3} \right)^2 + \left( \frac{\partial u_3}{\partial x_3} \right)^2 \right] = 0 \]

\[ 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} = 0 \]

\[ 2\varepsilon_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_3} = 0 \]

\[ 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \frac{\partial u_3}{\partial x_3} = 0 \]
\( u_0(x) \rightarrow u(x, t); w_0(x) \rightarrow w(x, t) \)

\( u_1(x, t) = u(x, t) - z \frac{\partial w_0}{\partial x}; u_2 = 0; u_3(x, t) = w(x, t) \)

\[ \varepsilon_{11} = \left( \frac{\partial u}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} \right) + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \]

\[ \sigma_{11} = E \varepsilon_{11} \]

\[ U = \frac{1}{2} \int_V E \varepsilon_{11}^2 dV \quad \& \quad T = \frac{1}{2} \int_V \rho \left( \dot{u}^2 + \dot{w}^2 \right) dV \]

\[ U = \frac{1}{2} \int \int_A \int E \left\{ \left( \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right\}^2 dA dx \]

\[ = \frac{1}{2} \int \int_A \int E \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + z^2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{4} \left( \frac{\partial w}{\partial x} \right)^4 - 2z \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 \right. \\
- \left. z \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial w}{\partial x} \right)^2 \right\} dA dx \]
\[ U = \frac{1}{2} \int_A \int_0^l E \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + z^2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{4} \left( \frac{\partial w}{\partial x} \right)^4 \right\} - 2z \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 \} dA dx \]

\[-z \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial w}{\partial x} \right)^2 \} dA dx\]

Consider beam to possess symmetric cross section.

\[ \Rightarrow U = \frac{1}{2} \int_0^l AE \left( \frac{\partial u}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^l EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx \]

\[ + \frac{1}{8} \int_0^l AE \left( \frac{\partial w}{\partial x} \right)^4 dx + \frac{1}{2} \int_0^l AE \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 dx \]

New terms due to presence of nonlinearity

\[ T = \frac{1}{2} \int_v \rho \left( \dot{u}^2 + \dot{w}^2 \right) dV = \frac{1}{2} \int_0^l m \left( \dot{u}^2 + \dot{w}^2 \right) dx \]
\[ L = \frac{1}{2} \int_{0}^{l} m(\ddot{u}^2 + \dot{w}^2) \, dx - \frac{1}{2} \int_{0}^{l} AE \left( \frac{\partial u}{\partial x} \right)^2 \, dx - \frac{1}{2} \int_{0}^{l} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx \]

\[ - \frac{1}{8} \int_{0}^{l} AE \left( \frac{\partial w}{\partial x} \right)^4 \, dx - \frac{1}{2} \int_{0}^{l} AE \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 \, dx \]
\[ u(x,t) = \sum_{i=5}^{6} u_i(t) \phi_i(x) \]

\[ w(x,t) = \sum_{i=1}^{4} u_i(t) \phi_i(x) \]

\[ \phi_1(x) = 1 - 3 \frac{x^2}{l^2} + 2 \frac{x^3}{l^3} \; ; \; \phi_2(x) = x - 2 \frac{x^2}{l} + \frac{x^3}{l^2} \; ; \]

\[ \phi_3(x) = 3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3} \; ; \; \phi_4(x) = - \frac{x^2}{l} + \frac{x^3}{l^2} \]

\[ \phi_5(x) = 1 - \frac{x}{l} \; ; \; \phi_6(x) = \frac{x}{l} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_i} \right) - \frac{\partial L}{\partial u_i} = 0, i = 1, 2, \cdots, 6 \]
\[ L = \frac{1}{2} \int_0^l m \left( \dot{u}^2 + \dot{w}^2 \right) dx - \frac{1}{2} \int_0^l AE \left( \frac{\partial u}{\partial x} \right)^2 dx - \frac{1}{2} \int_0^l EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx \]

Lead to the structural element mass and stiffness matrices deduced earlier

\[ -\frac{1}{8} \int_0^l AE \left( \frac{\partial w}{\partial x} \right)^4 dx - \frac{1}{2} \int_0^l AE \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 dx \]

Newer terms due to nonlinearity

Consider \( L_1 = \frac{1}{8} \int_0^l AE \left( \frac{\partial w}{\partial x} \right)^4 dx = \frac{1}{8} \int_0^l AE \left\{ \sum_{i=1}^4 u_i(t) \phi'_i(x) \right\}^4 dx \)

\[ \frac{\partial L_1}{\partial u_k} = \frac{1}{2} \int_0^l AE \left\{ \sum_{i=1}^4 u_i(t) \phi'_i(x) \right\}^3 \phi'_k(x) dx = \frac{1}{2} \sum_{i=1}^4 \sum_{j=1}^4 \sum_{m=1}^4 u_i(t) u_j(t) u_m(t) I_{ijmk} \]

\[ I_{ijmk} = \int_0^l \int_0^l \int_0^l AE \phi'_i(x) \phi'_j(x) \phi'_m(x) \phi'_k(x) dx; k = 1, 2, 3, 4 \]
Similarly, consider \( L_2 = \frac{1}{2} \int_0^l AE \frac{\partial u}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 \, dx \)

\[
L_2 = \frac{1}{2} \int_0^l AE \left( \sum_{i=5}^6 u_i(t) \phi_i'(x) \right) \left( \sum_{i=1}^4 u_i(t) \phi_i'(x) \right)^2 \, dx
\]

\[
\frac{\partial L_2}{\partial u_k} = \int_0^l AE \left( \sum_{i=5}^6 u_i(t) \phi_i'(x) \right) \left( \sum_{i=1}^4 u_i(t) \phi_i'(x) \right) \phi_k'(x) \, dx; \ k = 1, 2, 3, 4
\]

\[
= \sum_{i=5}^6 \sum_{j=1}^4 u_i(t) u_j(t) \int_0^l AE \phi_i'(x) \phi_j'(x) \phi_k'(x) \, dx; \ k = 1, 2, 3, 4
\]

\[
= \sum_{i=5}^6 \sum_{j=1}^4 u_i(t) u_j(t) J_{ijk} \ k = 1, 2, 3, 4
\]

\[
J_{ijk} = \int_0^l AE \phi_i'(x) \phi_j'(x) \phi_k'(x) \, dx; \ i = 5, 6; \ j, k = 1, 2, 3, 4
\]
\[ L_2 = \frac{1}{2} \int_0^l \left[ \frac{AE}{6} \left( \sum_{i=5}^6 u_i(t)\phi'_i(x) \right) \left( \sum_{i=1}^4 u_i(t)\phi'_i(x) \right) \right] dx \]

\[ \frac{\partial L_2}{\partial u_k} = \frac{1}{2} \int_0^l \left[ \frac{AE}{6} \left( \sum_{i=5}^6 u_i(t)\phi'_i(x) \right) \left( \sum_{i=1}^4 u_i(t)\phi'_i(x) \right) \right]^2 \phi'_k(x) dx; k = 5, 6 \]

\[ = \frac{1}{2} \int_0^l AE \sum_{i=5}^6 \sum_{r=1}^4 \sum_{s=1}^4 u_i(t)u_r(t)u_s(t)\phi'_i(x)\phi'_r(x)\phi'_s(x)\phi'_k(x) dx; k = 5, 6 \]

\[ = \frac{1}{2} \sum_{i=5}^6 \sum_{r=1}^4 \sum_{s=1}^4 u_i(t)u_r(t)u_s(t)K_{irsk} \]

\[ K_{irsk} = \int_0^l AE \phi'_i(x)\phi'_r(x)\phi'_s(x)\phi'_k(x) dx; r, s = 1, 2, 3, 4; i, k = 5, 6 \]
Form of the element equation of motion

\[
[M]_e \{\ddot{u}\}_e + [K]_e \{u\}_e + \{\text{Vector of quadratic and cubic terms in } u(t) \}_e = 0
\]

Remarks

- Assembly of element level matrices and vectors can be done as before to obtain the global equations of motion.
- Derivation of external forces and imposition of BCs again follows the earlier developed procedure.
- The resulting equations of motion would be of the form

\[
M\ddot{u} + C\dot{u} + Ku + g[u] = F(t); u(0) \& \dot{u}(0) \text{ specified.}
\]
- These equations can be integrated using numerical procedures discussed earlier (see Lecture 16).
Timoshenko beam element

\[ u_1 = u_0(x) + z\phi(x) \]
\[ u_2 = 0 \]
\[ u_3 = w_0(x) \]

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right] = \left( \frac{du_0}{dx} + z \frac{d\phi}{dx} \right) + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \]

\[ \varepsilon_{22} = 0 \]
\[ \varepsilon_{33} = 0 \]
\[ 2\varepsilon_{12} = 0 \]
\[ 2\varepsilon_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \phi(x) + \frac{dw_0}{dx} \]
\[ 2\varepsilon_{23} = 0 \]

\[ U = \frac{1}{2} \int_V \left( \sigma_{11} \varepsilon_{11} + \varepsilon_{13} \sigma_{13} \right) dV \]
How about a more general theory?

• Allow measures of strain and stress to be defined consistent with deformations.
• Allow for material nonlinearity
References


- http://www.colorado.edu/engineering/CAS/courses.d/NFEM.d/Home.html [Professor C A Felippa, University of Colorado at Boulder]

- W F Chen and D J Han, 2008, Plasticity for structural engineers, J.Ross publishing/Cengage Learning, New Delhi.
Notations

• Indicial notations
• Algebraic notations
• Matrix notations
• Tensor notations
Indicial notations

- A set of variables $x_1, x_2, \cdots, x_n$ is denoted as $x_i$.

The range of values taken by the index $i$, needs to be specified.

Typically, $i = 1, 2, 3$.

- Repeated indices implies summation

  $\alpha = \sum_{i=1}^{n} a_i x_i$ is written as $\alpha = a_i x_i$ ($i = 1, 2, \cdots, n$) (Note: $\alpha = a_i x_i = a_s x_s$)

  $U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} u_i u_j$ is written as $U = \frac{1}{2} k_{ij} u_i u_j$ ($i, j = 1, 2, \cdots, n$)

- Kronecker delta $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

- $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ is written as $ds^2 = \delta_{ij} dx_i dx_j$

- Permutation symbol $\varepsilon_{ijk}$

  $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

  $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

  $\varepsilon_{111} = \varepsilon_{222} = \varepsilon_{333} = \cdots = 0$
\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

\[ |A| = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3} \]

**ε - δ identity:** \[ \varepsilon_{ijk} \varepsilon_{rst} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks} \]

**Differentiation**

\[ f \equiv f (x_1, x_2, \ldots, x_n) \]

\[ df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n \]

is written as \[ df = \frac{\partial f}{\partial x_i} dx_i \] (\( i = 1, 2, \ldots, n \))

**The , symbol**

Consider

\[ f_1 \equiv f_1 (x_1, x_2, x_3), f_2 \equiv f_2 (x_1, x_2, x_3), f_3 \equiv f_3 (x_1, x_2, x_3) \]

\[ f_{i,j} = \frac{\partial f_i}{\partial x_j} \]

Similarly \[ \sigma_{ij,k} = \frac{\partial \sigma_{ij}}{\partial x_k} \]
Vectors and tensors are represented by single letters (bold face)

\[ x = (x_1, x_2, x_3) \]
\[ y = (y_1, y_2, y_3) \]
\[ x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 \]
\[ z = x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 z_1 + e_2 z_2 + e_3 z_3 \]
\[ z_i = \varepsilon_{ijk} x_j y_k \]

\[ \text{grad} = \nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i} \]

\[ f = f(x_1, x_2, x_3) = \text{scalar function} \]
\[ \text{grad } f = e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3} = e_1 f_1 + e_2 f_2 + e_3 f_3 \]
\[ f_i = \frac{\partial f}{\partial x_i} \]
\[ \nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i} \]

Consider a vector valued function \( F \)

\[ \text{div } F = \nabla \cdot F = \left( e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} \right) \cdot \left( e_1 F_1 + e_2 F_2 + e_3 F_3 \right) \]

\[ = \left( \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) = \frac{\partial F_i}{\partial x_i} \]

\[ z = \text{Curl } F = \nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = e_1 z_1 + e_2 z_2 + e_3 z_3 \]

\[ z_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \]
\[ \nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i} \]

Consider a vector valued function \( F \)

\[
\text{grad } F = (\nabla, F) = e_1 \frac{\partial F}{\partial x_1} + e_2 \frac{\partial F}{\partial x_2} + e_3 \frac{\partial F}{\partial x_3} \\
= e_1 \frac{\partial}{\partial x_1} \left( e_1 F_1 + e_2 F_2 + e_3 F_3 \right) + e_2 \frac{\partial}{\partial x_2} \left( e_1 F_1 + e_2 F_2 + e_3 F_3 \right) + e_3 \frac{\partial}{\partial x_3} \left( e_1 F_1 + e_2 F_2 + e_3 F_3 \right)
\]

Thus \( \text{grad } F \) can be described by the following matrix

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_3}{\partial x_1} \\
\frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_3}{\partial x_2} \\
\frac{\partial F_1}{\partial x_3} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_3}{\partial x_3}
\end{bmatrix}
\]
Matrix notations

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad r^2 = x'x; \quad \sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}; \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix} \]

\[ U = \frac{1}{2} \sigma^t \varepsilon = \frac{1}{2} \varepsilon^t C \varepsilon \]

No explicit mention of connective symbols (multiplication)

Tensor notations

• indices not shown

• applicable to Cartesian and other coordinate systems

\[ x_i y_i = x.y \quad (\text{dot denotes contraction of inner indices}) \]

\[ A_{ij} B_{ij} = A : B \quad (\text{colon denotes contraction of a pair of repeated indices}) \]

\[ \sigma_{ij} = c_{ijkl} \varepsilon_{kl} \equiv \sigma = C : \varepsilon \]
\[ \phi . K . \phi = \phi' K \phi = \phi_i K_{ij} \phi_j \]

Tensor Matrix Indicial

\[ \frac{1}{2} \varepsilon : C : \varepsilon = \frac{1}{2} \varepsilon^t C \varepsilon = \frac{1}{2} \varepsilon_i C_{ij} \varepsilon_j \]

Tensor Matrix Indicial

\[ \sigma' = C \sigma C^t \quad \sigma'_{ij} = C_{im} \sigma_{mn} C_{jn} \]

Matrix Indicial

\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0 \]

\[ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 = 0 \]

\[ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0 \]

Indicial Matrix Full notation Notation of last resort
Continuum hypothesis

• Matter is infinitely divisible.
• Each infinitesimal element retains all the properties of the material.
• Newtonian mechanics is directly applicable.
• Calculus works (governing equations can be derived as PDE-s or ODE-s; variational approaches can be adopted to describe system behavior)
• Attention is limited to characteristic dimensions > about $10^{-6}$ cm (diameter of a water molecule $\approx 10^{-8}$ cm)
• The theory is valid for both solids and fluids
• Notions of density, temperature, pressure, etc., at a point make sense.
• Primary aim: to model macroscopic behavior of solids and fluids.

• Ignores the atomic structure of matter.
  (Matter consists of discrete particles which are perpetually in motion)
• Questions on treatment of molecular, grain or crystal structure are not addressed.
Three themes

- Kinematics: Motion and deformation
- Kinetics: Concept of stress
- Balance laws (common to fluids and solids)

Understanding of

- rotations
- alternative definitions of stress and strain
- treatment of material nonlinear behavior
Kinematics:
Study of motion and deformation without concerning with causes of motion and deformation.

Body B at time=0

\[ x = \phi(X, t) \]
Reference frame: origin O; orthonormal basis: $e_1, e_2, e_3$.

Body B occupies different regions $\Omega_0, \cdots, \Omega$ at time instants 0, $\cdots$, $t$.

The regions $\Omega_0, \cdots, \Omega$ occupied by the body at different time instants 0, $\cdots$, $t$ are known as configurations of the body at the respective time instants.

**time** $t = 0$

$\Omega_0 = \text{initial state of the body; initial configuration.}$

Could also be taken as the reference configuration with respect to which motion is described.

**Undeformed** configuration: is an idealization.

$\Gamma_0 = \text{boundary of the initial configuration.}$

**time** $t$

$\Omega = \text{current state of the body; current (deformed) configuration.}$

$\Gamma_1 = \text{boundary of the current configuration.}$
Eulerian (spatial) and Lagrangian (material) coordinates

In Lagrangian description we take \((X_1, X_2, X_3, t)\) as independent variables.

In Eulerian description we take \((x_1, x_2, x_3, t)\) as independent variables.

\[
X = X_i e_i \quad (\equiv X_1 e_1 + X_2 e_2 + X_3 e_3)
\]

- Position vector of a material point in the initial configuration.
- This does not change with time.
- Labels all material points.

\[
x = x_i e_i \quad (\equiv x_1 e_1 + x_2 e_2 + x_3 e_3)
\]

- Provides the position of a point in the current configuration.
- Changes as configurations evolve in time.

In problems of solid mechanics, Lagrangian description is often used. Lagrangian description is also known as material description and Eulerian description is also known as spatial description.
Motion

\[ x_1 = \phi_1(X_1, X_2, X_3, t) \]
\[ x_i = \phi_i(X, t), i = 1, 2, 3 \Rightarrow x_2 = \phi_2(X_1, X_2, X_3, t) \]
\[ x_3 = \phi_3(X_1, X_2, X_3, t) \]

When reference and initial configurations coincide we get
\[ x(X, 0) = X = \phi(X, 0) \]
\[ X_i = x_i(X, 0) = \phi_i(X, 0) \Rightarrow \phi(X, 0) \text{ is an identity transformation.} \]

Material coordinates

**Displacement**: \( u(X, t) = x - X = \phi(X, t) - \phi(X, 0) = \phi(X, t) - X \)

**Velocity**: \( v(X, t) = \frac{\partial \phi(X, t)}{\partial t} = \frac{\partial u(X, t)}{\partial t} \equiv \ddot{u} \)

**Acceleration**: \( a(X, t) = \frac{\partial v(X, t)}{\partial t} = \frac{\partial^2 u(X, t)}{\partial t^2} \equiv \ddot{u} \)
Spatial coordinates

\[ \frac{D}{Dt} \phi(x,t) = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_j} \frac{dx_j}{dt} \]

\[ a_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \frac{dx_j}{dt} = \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{dx_j}{dt}; i = 1, 2, 3 \]

\[ a = \frac{\partial v}{\partial t} + v \cdot \text{grad } v \]
Deformation gradient

Body B at time=0

\[ x = \phi(X, t) \]

Body B at time=t

\[ \Omega_0 \]

Body B at time=0

\[ \Omega \]

\[ \Omega_0 \]

\[ \Gamma_0 \]

\[ X_P \]

\[ Q \]

\[ dX \]

\[ P \]

\[ u_Q \]

\[ u_P \]

\[ x_3, X_3 \]

\[ e_3 \]

\[ x_1, X_1 \]

\[ e_1 \]

\[ e_2 \]

\[ x_2, X_2 \]

\[ x_p \]

\[ x_q \]

\[ p \]

\[ q \]

\[ dx \]

\[ dX \]