Finite element method for structural dynamic and stability analyses

Module-9

Structural stability analysis

Lecture-32 Dynamic analysis of stability and analysis of time varying systems

Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India
3D beam element

Is it that computational effort increases and we need to handle larger sized matrices? OR

Are there any new phenomena that we need to be concerned about?

3D beam element: a simplified model

Effects included: coupling between axial forces with bending along two axes, torsion about longitudinal axis

\[ u(x,t), v(x,t), w(x,t), \theta(x,t) \]

\[ E, G, I_x, I_y, I_z, \rho, A, J, l \]
Geometric stiffness matrix for a thin plate bending element
(Kirchoff-Loeve plate)

Membrane forces

\[ N_x = \int_{-t/2}^{t/2} \sigma_{xx} \, dz; \quad N_y = \int_{-t/2}^{t/2} \sigma_{yy} \, dz; \quad N_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} \, dz \]
Imperfection sensitive structures

\[ P_m(\varepsilon) \]

\[ \varepsilon \]

\[ P \]

\[ k_1 \]

\[ l \]

\[ k_2 \]

\[ k_3 \]

\[ P \]

\[ A \]

\[ B \]

\[ \alpha \]

\[ \varepsilon = \frac{\pi}{50} \]

\[ \varepsilon = 0 \]

\[ k = P l \cos \theta > 0 \]

Black: stable
Red: Unstable

Stable if

\[ k = P l \cos \theta > 0 \]
Two hinged circular arch under uniform pressure.
Would not have created BM in pre-buckled state.
What is critical value of $q$?
That is, what is the value of $q$, which induces bending in the arch?

Timoshenko and Gere, p207
\[
\frac{d^2 w}{d\theta^2} + w = -\frac{MR^2}{EI} = -\frac{SwR^2}{EI} = -\frac{(qR)wR^2}{EI}
\]

\[
\frac{d^2 w}{d\theta^2} + w \left(1 + \frac{qR^3}{EI}\right) = 0 \text{ with BCS: } w(0) = 0, w(2\alpha) = 0
\]

\[
\frac{d^2 w}{d\theta^2} + k^2 w = 0 \text{ with } k^2 = \left(1 + \frac{qR^3}{EI}\right)
\]

\[
w(\theta) = A \cos k\theta + B \sin k\theta
\]

\[
w(0) = 0 \Rightarrow A = 0
\]

\[
w(2\alpha) = 0 \Rightarrow \sin 2k\alpha = 0 \Rightarrow 2k\alpha = n\pi, n = 1, 2, \Rightarrow k = \frac{\pi}{\alpha}
\]

\[
\Rightarrow \left(1 + \frac{qR^3}{EI}\right) = \frac{\pi^2}{\alpha^2} \Rightarrow q_c = \frac{EI}{R^3} \left(\frac{\pi^2}{\alpha^2} - 1\right)
\]

Exercise: Using Euler-Bernoulli beam elements, verify this result.
Dynamic analysis of stability and analysis of time varying systems

Topics

- Dynamic analysis of structures in presence of static initial stresses
- Parametrically excited systems: stability analysis
- Statically loaded structures which require dynamic considerations to establish stability
- Response analysis of time varying systems
Dynamic analysis of a beam column

\[ f(x,t) \]

\[ EI \frac{\partial^4 y}{\partial x^4} + P \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} = f(x,t) \]

BCs: \( y(0,t) = 0, \) \( EI \frac{\partial^2 y}{\partial x^2}(0,t) = 0, \) \( y(l,t) = 0, \) \( EI \frac{\partial^2 y}{\partial x^2}(l,t) = 0 \)

ICs: \( y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = v_0(x) \)
\[
EI \frac{\partial^4 y}{\partial x^4} + P \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} = f(x, t) \equiv EI y'''' + Py'' + m\ddot{y} + c\dot{y} = f(x, t)
\]

Undamped free vibration analysis

\[
EI \frac{\partial^4 y}{\partial x^4} + P \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = 0
\]

\[y(x, t) = \phi(x) \exp(i\omega t)\]

\[\Rightarrow \{EI \phi'''' + P\phi'' - m\omega^2 \phi\} \exp(i\omega t) = 0 \Rightarrow EI \phi'''' + P\phi'' - m\omega^2 \phi = 0\]

\[\phi'''' + \alpha^2 \phi'' - \lambda^4 \phi = 0 \text{ with } \alpha^2 = \frac{P}{EI} \text{ and } \lambda^4 = \frac{m\omega^2}{EI}\]

This constitutes an eigenvalue problem. Let us seek solution in the form

\[\phi(x) = \phi_0 \exp(sx)\]

\[\Rightarrow s^4 + \alpha^2 s^2 - \lambda^4 = 0\]

\[\Rightarrow s^2 = \frac{-\alpha^2 \pm \sqrt{\alpha^4 + 4\lambda^4}}{2}\]
\[ s^2 = \frac{-\alpha^2 \pm \sqrt{\alpha^4 + 4\lambda^4}}{2} \]

\[ s_1^2 = \frac{-\alpha^2 + \sqrt{\alpha^4 + 4\lambda^4}}{2} > 0 \Rightarrow (s_1)_{1,2} = \pm \epsilon \]

\[ s_2^2 = \frac{-\alpha^2 - \sqrt{\alpha^4 + 4\lambda^4}}{2} < 0 \Rightarrow (s_2)_{1,2} = \pm i\delta \]

\[ \phi(x) = A \cosh \epsilon x + B \sinh \epsilon x + C \sin \delta x + D \cos \delta x \]

\[ \phi'(x) = A \epsilon \sinh \epsilon x + B \epsilon \cosh \epsilon x + C \delta \cos \delta x - D \delta \sin \delta x \]

\[ \phi''(x) = A \epsilon^2 \cosh \epsilon x + B \epsilon^2 \sinh \epsilon x - C \delta^2 \sin \delta x - D \delta^2 \cos \delta x \]

\[ \phi(0) = 0 \Rightarrow A + D = 0 \]

\[ \phi''(0) = 0 \Rightarrow A \epsilon^2 - D \delta^2 = 0 \]

\[ \phi(l) = 0 \Rightarrow B \sinh \epsilon l + C \sin \delta l = 0 \]

\[ \phi''(l) = 0 \Rightarrow B \epsilon^2 \sinh \epsilon l - C \delta^2 \sin \delta l = 0 \]
\[
\begin{bmatrix}
\sinh \varepsilon l & \sin \delta l \\
\varepsilon^2 \sinh \varepsilon l & -\delta^2 \sin \delta l
\end{bmatrix}
\begin{bmatrix}
B \\
C
\end{bmatrix} = 0
\]

For nontrivial solutions, \[
\begin{vmatrix}
\sinh \varepsilon l & \sin \delta l \\
\varepsilon^2 \sinh \varepsilon l & -\delta^2 \sin \delta l
\end{vmatrix} = 0
\]

\[
\Rightarrow -\left(\delta^2 + \varepsilon^2\right) \sinh \varepsilon l \sin \delta l = 0
\]

\(\left(\delta^2 + \varepsilon^2\right) \neq 0\)

\[
\sinh \varepsilon l = 0 \Rightarrow \varepsilon = 0 \Rightarrow \alpha^2 = \sqrt{\alpha^4 + 4\lambda^4} \Rightarrow \lambda = 0 \text{ Not ok.}
\]

\[
\Rightarrow \sin \delta l = 0 \Rightarrow \delta_n l = n\pi, n = 1, 2, \ldots
\]

\[
\Rightarrow \phi_n(x) = \phi_{0n} \sin \frac{n\pi x}{l}
\]

\[
s^4 + \alpha^2 s^2 - \lambda^4 = 0 \quad \& \quad s_2^2 = -\delta^2 \Rightarrow
\]

\[
\delta^4 - \delta^2 \alpha^2 - \lambda^4 = 0
\]

\[
\left(\frac{n\pi}{l}\right)^4 - \left(\frac{n\pi}{l}\right)^2 \alpha^2 - \lambda^4 = 0
\]
\[
\left( \frac{n\pi}{l} \right)^4 - \left( \frac{n\pi}{l} \right)^2 \alpha^2 - \lambda^4 = 0 \Rightarrow \left( \frac{n\pi}{l} \right)^4 - \left( \frac{n\pi}{l} \right)^2 \frac{P}{EI} = \frac{m\omega_n^2}{EI}
\]

\[
\omega_n^2 = \frac{EI}{m} \left( \frac{n\pi}{l} \right)^4 \left[ 1 - \frac{P}{EI} \frac{l^2}{n^2\pi^2} \right]
\]

\[
\omega_n = \frac{n^2\pi^2}{l^2} \sqrt{\frac{EI}{m}} \left[ 1 - \frac{P}{P_{ncr}} \right]^{1/2} = \omega_{n0} f(P)
\]
Remarks
• Due to presence of $P$, the natural frequencies get lowered. This has important implications in characterizing resonant characteristics of the system.
• In this case, it is observed that the mode shapes are not affected by $P$. This may not be true in general.
• At $P = P_{cr}$, $\omega_1 = 0$
• Orthogonality relations

$$\int_0^l m(x) \phi_n(x) \phi_k(x) \, dx = 0, \text{ for } n \neq k$$

$$\int_0^l \left[ EI(x) \phi_n''(x) \phi_k''(x) + P \phi_n'(x) \phi_k'(x) \right] \, dx = 0, \text{ for } n \neq k$$
Forced response analysis

\(EI \frac{\partial^4 y}{\partial x^4} + P \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} = f(x, t)\)

\(y(x, t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x)\)

\[\Rightarrow \ddot{a}_n + 2\eta_n \omega_n \dot{a}_n + \omega_n^2 a_n = q_n(t)\]

Special case: \(c = 0, f(x, t) = 0 \Rightarrow \ddot{a}_n + \omega_n^2 a_n = 0\)

- \(P < P_{cr} \Rightarrow a_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t\)
- \(P = P_{cr} \Rightarrow a_n(t) = A_n + B_n t\)

Motion remains bounded for all times for all initial conditions

- \(P = P_{cr} \Rightarrow a_n(t) = A_n + B_n t\)

Motion grows linearly in time even for small initial conditions
\[ P > P_{cr} \implies a_n(t) = A_n \exp(\gamma_n t) + B_n \exp(-\gamma_n t) \quad \text{with} \quad \gamma_n^2 = -\omega_n^2 \]

Motion grows exponentially in time even for small initial conditions.

Presence of damping

\[ P < P_{cr} \implies \text{motion decays exponentially in time} \]

\[ P \geq P_{cr} \implies \text{no qualitative change in the behavior} \]
\[
\frac{ml}{420} \begin{bmatrix}
156 & 22l & 54 & -13l \\
22l & 4l^2 & 13l & -3l^2 \\
54 & 13l & 156 & -22l \\
-13l & -3l^2 & -22l & 4l^2
\end{bmatrix} \begin{bmatrix}
\ddot{u}_4
\end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix} \begin{bmatrix}
0
\end{bmatrix} = 0
\]

\[
\frac{P}{30l} \begin{bmatrix}
36 & 3l & -36 & 3l \\
-3l & 4l^2 & -3l & -l^2 \\
-36 & -3l & 36 & -3l \\
3l & -l^2 & -3l & 4l^2
\end{bmatrix} \begin{bmatrix}
0
\end{bmatrix} = 0
\]

\[
\frac{ml}{420} 4l^2 \dddot{u}_4 + \left( \frac{EI}{l^3} 4l^2 - \frac{P}{30l} 4l^2 \right) u_4 = 0 \Rightarrow \dddot{u}_4 + 420 \left( \frac{EI}{ml} - \frac{P}{30l} \right) u_4 = 0
\]
\[
\frac{ml}{420} \begin{bmatrix}
156 & 22l & 54 & -13l \\
22l & 4l^2 & 13l & -3l^2 \\
54 & 13l & 156 & -22l \\
-13l & -3l^2 & -22l & 4l^2
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_2 \\
\ddot{u}_4
\end{bmatrix}
+ \frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix}
\begin{bmatrix}
0 \\
u_2 \\
u_4
\end{bmatrix}
= 0
\]

\[
- \frac{P}{30l} \begin{bmatrix}
36 & 3l & -36 & 3l \\
-3l & 4l^2 & -3l & -l^2 \\
-36 & -3l & 36 & -3l \\
3l & -l^2 & -3l & 4l^2
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u_4
\end{bmatrix}
= 0
\]
Exercise

Plot the first natural frequency as a function of $P$

$2l = 4m$  
$2$  
$K$  
$l = 2m$  
$3$

$EI = 10^8 \text{ Nm}^2$, $l = 2m$, $m = 100 \text{ kg/m}$, $\frac{l^3 K}{EI} = 5$

$3P$
Parametrically excited systems

\[ EI \frac{\partial^4 y}{\partial x^4} + P(t) \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} = f(x,t) \]

BCs: \( y(0,t) = 0, \quad EI \frac{\partial^2 y}{\partial x^2}(0,t) = 0, \quad y(l,t) = 0, \quad EI \frac{\partial^2 y}{\partial x^2}(l,t) = 0 \)

ICs: \( y(x,0) = y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = v_0(x) \)

\[ P(t) = \text{Parametric excitation} \]

\[ f(x,t) = \text{External excitation} \]
Let \( f(x,t) = 0 \), & \( P(t) = P_0 \cos \Omega t \)

\[ \Omega = 0, P_{cr} = \frac{\pi^2 EI}{l^2} \]

\( \Omega \neq 0, P_0 < P_{cr} \), one would expect the beam to oscillate axially. However, for certain values of \( \Omega \), the beam can get into large amplitude bending oscillations, even for small values of \( P_0 \) (Parametric resonance). Note that when such oscillations occur, \( \Omega \) need not coincide with any of the bending/axial natural frequencies of the beam.

Under this condition, the amplitudes can grow exponentially. Conversely, if \( P_0 > P_{cr} \), the structure can remain stable for certain values of \( \Omega \).

\[
P(t) = \text{External excitations for axial oscillations} = \text{Parametric excitations for bending oscillations}
\]
A stack under bi-axial ground motion

\[ B M \text{ at } A = \int_{y}^{h} m\left[ g + \ddot{y}_g(t) \right]\left[ u(\xi, t) - u(y, t) \right] \, d\xi \]
BM at A = \int_{y}^{h} m \left[ g + \ddot{y}_g (t) \right] \left[ u(\xi,t) - u(y,t) \right] d\xi

\frac{\partial^2}{\partial y^2} \left[ EI \frac{\partial^2 u}{\partial y^2} + \int_{y}^{h} m \left[ g + \ddot{y}_g (t) \right] \left[ u(\xi,t) - u(y,t) \right] d\xi \right] + m \frac{\partial^2 u}{\partial t^2}

+ c \left( \frac{\partial u}{\partial t} - \dot{x}_g (t) \right) = f (x,t)

u(0,t) = x_g (t); u'(0,t) = 0

EIu'' (h,t) = 0; (EIu'')' (h,t) = 0

\left( EIu'' \right)'' + m \left[ g + \ddot{y}_g (t) \right] \left[ u' - (h - y)u'' \right] + m\ddot{u} + c \left( \ddot{u} - \dot{x}_g \right) = f (x,t)

Parametrically excited system under time dependent boundary conditions.
\begin{align*}
(Elu'')'' + m\left[ g + \ddot{y}_g(t) \right] [u' - (h - y)u''] + m\ddot{u} + c(u' - \dot{x}_g) &= f(x, t) \\
u(0, t) = x_g(t); u'(0, t) = 0 \\
Elu''(h, t) = 0; (Elu'')' (h, t) = 0 \\
w(y, t) &= u(y, t) - x_g(t) \\
w(y, t) &= \sum_{n=1}^{N} a_n(t) \phi_n(x) \\
\ddot{a}_n + 2\eta_n \omega_n \dot{a}_n + \omega_n^2 a_n + \frac{g + \ddot{y}_g(t)}{h} \sum_{n=1}^{N} \alpha_{nj} a_j(t) &= -\gamma_n \ddot{x}_g(t) \\
M\ddot{X} + C\dot{X} + \left[ K + K_0(t) \right] X &= Q(t) \\
X(0) = X_0; \dot{X}(0) = \dot{X}_0
\end{align*}
\[ M\ddot{X} + C\dot{X} + \left[ K + K_0(t) \right] X = Q(t) \]
\[ X(0) = X_0; \dot{X}(0) = \dot{X}_0 \]

- Coupled mdof system with time dependent stiffness matrix
- Notion of natural frequencies and mode shapes not valid
- Normal modes (with ground motions=0) would not uncouple

Questions
How to solve these equations?
Are there any new phenomenological features associated with the response?
What happens to the notion of resonance?
What happens to the notion of instability?
Structures under moving loads

Vehicle as a moving force

\[ m_v g \rightarrow a, v \]

\[ EIy^{iv} + m\ddot{y} + c\dot{y} = m_v g \delta \left(x - vt - \frac{1}{2}at^2\right) \text{ for } 0 < t < t_{exit} \]

\[ = 0 \text{ for } 0 < t < t_{exit} \]

\[ y(x,t) = \sum_{n=1}^{N} a_n(t) \phi_n(x) \]

\[ \Rightarrow \ddot{a}_n + 2\eta_n \omega_n \dot{a}_n + \omega_n^2 a_n = m_v g \phi_n \left(vt - \frac{1}{2}at^2\right) \text{ for } 0 < t < t_{exit} \]

\[ = 0 \text{ for } 0 < t < t_{exit} \]
\[ \ddot{a}_n + 2\eta_n \omega_n \dot{a}_n + \omega_n^2 a_n = m_v g \phi_n \left( vt - \frac{1}{2}at^2 \right) \quad \text{for } 0 < t < t_{exit} \]

\[ = 0 \quad \text{for } 0 < t < t_{exit} \]

\[ a = 0, \phi_n(x) = \phi_{n0} \sin \left( \frac{n\pi x}{l} \right) \]

\[ \ddot{a}_n + 2\eta_n \omega_n \dot{a}_n + \omega_n^2 a_n = m_v g \phi_{n0} \sin \left( \frac{n\pi vt}{l} \right) \quad \text{for } 0 < t < t_{exit} \]

\[ = 0 \quad \text{for } 0 < t < t_{exit} \]

Critical condition: \( \omega_1 = \frac{\pi v}{l} \)

\[ \Rightarrow \text{Critical velocity } v_c = \frac{\omega_1 l}{\pi} = \frac{l}{\pi} \frac{\pi^2}{l^2} \sqrt{\frac{EI}{m}} = \frac{\pi}{l} \sqrt{\frac{EI}{m}} \]

- Resonant conditions can prevail if a series of loads pass at velocity = \( v_c \)
Vehicle as a moving mass

\[ m_v \rightarrow a, v \]

\[
\begin{align*}
\frac{D}{Dt} y[x(t), t] &= \frac{\partial y}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} = \left( v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) y \\
\frac{D^2}{Dt^2} y[x(t), t] &= \left( v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \left( v \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right) = v^2 \frac{\partial^2 y}{\partial x^2} + 2v \frac{\partial^2 y}{\partial x \partial t} + a \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2}
\end{align*}
\]

\[ m_v g - m_v \frac{D^2 y}{Dt^2} \]
\( E I y'' + m \ddot{y} + c \dot{y} = \left\{ m_v g - m_v \frac{D^2 y}{Dt^2} \right\} \delta \left( x - vt - \frac{1}{2} at^2 \right) \) for \( 0 < t < t_{exit} \)
\[
= 0 \text{ for } 0 < t < t_{exit}
\]
\[
\frac{D^2}{Dt^2} y \left[ x(t), t \right] = v^2 \frac{\partial^2 y}{\partial x^2} + 2v \frac{\partial^2 y}{\partial x \partial t} + a \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2}
\]
For \( 0 < t < t_{exit} \)
\[
E I y'' + m \ddot{y} + m_v \frac{D^2 y}{Dt^2} \delta \left( x - vt - \frac{1}{2} at^2 \right) + c \dot{y} = m_v g \delta \left( x - vt - \frac{1}{2} at^2 \right)
\]
\[
E I y'' + m \ddot{y} + m_v \left\{ v^2 \frac{\partial^2 y}{\partial x^2} + 2v \frac{\partial^2 y}{\partial x \partial t} + a \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2} \right\} \delta \left( x - vt - \frac{1}{2} at^2 \right) + c \dot{y}
\]
\[
= m_v g \delta \left( x - vt - \frac{1}{2} at^2 \right)
\]
\[
\Rightarrow
M(t) \dddot{X} + C(t) \ddot{X} + K(t) X = F(t); X(0) = X_0; \dot{X}(0) = \dot{X}_0
\]
Vehicle as a moving sdof oscillator

\[ m_u = \text{unsprung mass} \]
\[ m_s = \text{sprung mass} \]
for $0 < t < t_{\text{exit}}$

\[ m_s \ddot{u} + c_v \left\{ \ddot{u} - \frac{D}{Dt} y \left[ x(t), t \right] \right\} + k_v \left\{ u - y \left[ x(t), t \right] \right\} = 0 \]

\[ EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t) \delta \left( x - vt - \frac{1}{2} at^2 \right) \]

\[ f(x,t) = (m_u + m_s) g + k_v \left\{ u - y \left[ x(t), t \right] \right\} + c_v \left\{ \ddot{u} - \frac{D}{Dt} y \left[ x(t), t \right] \right\} \]

\[ -m_u \frac{D^2}{Dt^2} y \left[ x(t), t \right] \]

\[ f(x,t) = \text{wheel force} \]

\[ \Rightarrow \]

\[ M(t) \dddot{X} + C(t) \ddot{X} + K(t) X = F(t); X(0) = X_0; \dot{X}(0) = \dot{X}_0 \]
Periodic loading applied symmetrically on an arch generally is expected to produce symmetric oscillations of the arch. Under certain conditions, however, the arch can undergo asymmetric oscillations of large amplitudes.

Periodic $P(t)$ generally expected to produce axial deformations. Under certain conditions the ring can undergo bending oscillations of large amplitude.
Periodic $P(t)$ generally expected to produce bending deformations in the plane of loading. Under certain conditions the beam can undergo torsional oscillations of large amplitude.
Stability of steady state motions

\[ m\ddot{x} + c\dot{x} + kx + \alpha x^3 = P \cos \Omega t; \quad x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \]

\[ \lim_{t \to \infty} x(t) = X \cos (3\Omega t - \theta) = x^*(t) \]

\[ x(t) \to x^*(t) + \Delta(t) \quad \left[ \Delta(t) \text{ is "small"} \right] \]

\[ \Rightarrow \]

\[ m\ddot{\Delta} + c\Delta + k\Delta + 3\alpha x^*^2 (t) \Delta = 0 \]

\[ \lim_{t \to \infty} |\Delta(t)| \to ? \]

\[ x^*^2 (t) : \text{acts as a parametric excitation} \]
Line of action of $P$ remains unaltered as beam deforms. Static analysis can be used to find $P_{cr}$.

Line of action of $P$ remains tangential to the deformed beam axis. Static analysis does not lead to correct value of $P_{cr}$. “Follower” forces.
Problem 1
How to characterize resonances in systems governed by equations of the form
\[ M(t) \ddot{X} + C(t) \dot{X} + K(t) X = 0; \quad X(0) = X_0; \quad \dot{X}(0) = \dot{X}_0 \]
when the parametric excitations are periodic.

Problem 2
How to arrive at FE models for PDE-s with time varying coefficients?

Problem 3
Are there any situations in statically loaded systems, wherein one needs to use dynamic analysis to infer stability conditions?
Qualitative analysis of parametrically excited systems

\[ \ddot{u}(t) + p_1(t)\dot{u}(t) + p_2(t)u(t) = 0 \]

\[ u(0) = u_0; \dot{u}(0) = \dot{u}_0 \]

\[ p_i(t+T) = p_i(t), i = 1, 2 \]

The governing equation is a linear second order ODE with time varying coefficients. It admits two fundamental solutions.

Digress

\[ \ddot{x} + \omega^2 x = 0 \]

\[ x(t) = a \cos \omega t + \bar{b} \sin \omega t = ax_1(t) + bx_2(t) \]

with \( x_1(t) = \cos \omega t \), \( x_2(t) = \frac{\sin \omega t}{\omega} \)

Notice: \( x_1(0) = 1 \& \dot{x}_1(0) = 0 \)

\[ x_2(0) = 0 \& \dot{x}_2(0) = 1 \]

Any solution can be expressed as linear combination of \( x_1(t) \& x_2(t) \).
\[ \ddot{u}(t) + p_1(t) \dot{u}(t) + p_2(t) u(t) = 0 \]

Let \( u_1(t) \) and \( u_2(t) \) be the fundamental solutions of this equation.

\[ \Rightarrow u(t) = c_1 u_1(t) + c_2 u_2(t) \]

Consider the governing equation at \( t + T \)

\[ \ddot{u}(t + T) + p_1(t + T) \dot{u}(t + T) + p_2(t + T) u(t + T) = 0 \]

Since \( p_i(t + T) = p_i(t), i = 1, 2 \), we get

\[ \ddot{u}(t + T) + p_1(t) \dot{u}(t + T) + p_2(t) u(t + T) = 0 \]

\[ \Rightarrow \text{If } u(t) \text{ is a solution } \Rightarrow u(t + T) \text{ is also a solution.} \]

\[ \Rightarrow \]

\[ u_1(t + T) = a_{11} u_1(t) + a_{12} u_2(t) \]

\[ u_2(t + T) = a_{21} u_1(t) + a_{22} u_2(t) \]

\[ \Rightarrow \{u(t + T)\} = [A]\{u(t)\} \]

We are interested in nature of the solution as \( t \to \infty \).
\[ \{u(t+T)\} = [A]\{u(t)\} \]

\[ \lim_{t \to \infty} u(t) \to ? \]

This is equivalent to asking \( \lim_{n \to \infty} u(t+nT) \to ? \)

\[ u(t+T) = Au(t) \]

\[ u(t+2T) = Au(t+T) = A^2u(t) \]

\[ \vdots \]

\[ u(t+nT) = Au(t+(n-1)T) = A^nu(t) \]

\[ \Rightarrow \] The behavior of \( \lim_{n \to \infty} u(t+nT) \) is controlled by the behavior of \( \lim_{n \to \infty} A^n \).

Intutively, one can see that this, in turn, depends upon the nature of eigenvalues of \( A \).
Digress:
Consider the scalar case of
\[ x(t + T) = \alpha x(t) \]
\[ \Rightarrow x(t + nT) = \alpha^n x(t) \]

- \( \lim_{n \to \infty} x(t + nT) \to 0 \) if \( |\alpha| < 1 \)
- \( \lim_{n \to \infty} x(t + nT) \to \infty \) if \( |\alpha| > 1 \)
- \( x(t + T) = x(t) \Rightarrow x(t) \) is periodic with period \( T \) if \( \alpha = 1 \)
- \( x(t + 2T) = x(t) \Rightarrow x(t) \) is periodic with period \( 2T \) if \( \alpha = -1 \)

Similar situation can be expected to prevail in analysing \( u(t + T) = Au(t) \)

\[ x_1 = -x_1 \]
\[ x_3 = -x_2 = x_1 \]