Finite element method for structural dynamic and stability analyses

Module-6

Model reduction and substructuring schemes

Lecture-17 Model reduction schemes

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Model reduction and substructuring techniques

The need:

• Treatment of large scale problems

• Dealing with situations when the results from experiments need to be discussed in conjunction with predictions from mathematical models.
  - Mismatch of dof-s in measurement and computational models.

• Different parts of a structure are developed by different teams (possibly by using both experimental and computational tools) and model for the built-up structure needs to be developed in terms of constituent 'substructures'.

• Hybrid simulations: here we combine both experimental and computational models for the same structure. A part of the structure is studied experimentally and a part of the structure computationally.
Problem of model reduction

Consider a $N$ dof FE model for a linear system governed by

$$M\ddot{X} + C\dot{X} + KX = F(t); X(0) = X_0 \& \dot{X}(0) = \dot{X}_0$$

The objective of model reduction is to replace the above $N$ dof system by an equivalent $n$ dof system ($n < N$)

$$M_r\ddot{X}_m + C_r\dot{X}_m + K_rX_m = F_r(t)$$

$M_r, C_r, K_r =$ Reduced $n \times n$ structural matrices.

$X_m = n \times 1$ vector of dof-s which has been retained in the reduced model.

In all the model reduction techniques, the displacement vector $X(t)$ is taken to be of the form

$$X(t) = \begin{cases} X_m(t) \\ X_s(t) \end{cases}; \quad X_m(t): n \times 1 \text{ master dof-s}$$

$$X_s(t): (N - n) \times 1 \text{ slave dof-s}$$
We represent

\[ X(t) = \begin{bmatrix} X_m(t) \\ X_s(t) \end{bmatrix} = \Psi X_m(t) \]

where \( \Psi \) is an \( N \times n \) transformation matrix.

\[ \Rightarrow \]

\[ M \Psi \ddot{X}_m(t) + C \Psi \dot{X}_m(t) + K \Psi X_m(t) = F(t) \]

\[ \Rightarrow \]

\[ \Psi^t M \Psi \ddot{X}_m(t) + \Psi^t C \Psi \dot{X}_m(t) + \Psi^t K \Psi X_m(t) = \Psi^t F(t) \]

\[ \Rightarrow \]

\[ M_r \ddot{X}_m + C_r \dot{X}_m + K_r X_m = F_r(t) \]

\[ M_r = \Psi^t M \Psi = \text{Reduced mass matrix}; \ M_r^t = M_r \]

\[ C_r = \Psi^t C \Psi = \text{Reduced damping matrix}; \ C_r^t = C_r \]

\[ K_r = \Psi^t K \Psi = \text{Reduced stiffness matrix}; \ K_r^t = K_r \]

\[ F_r(t) = \Psi^t F(t) = \text{Reduced force vector} \]

**Question:** how to select \( \Psi \)?
Question: how to select \( \Psi \)?

The original model would have \( N \)-pairs of natural frequencies and eigenvectors. The reduced model would have only \( n \)-eigenpairs. Should these be equivalent?

Should the FRF-s over a given frequency range of the reduced system serve as acceptable approximations to the corresponding FRF-s of the original system?

Similarly, should transient response to dynamic excitations for the reduced system serve as acceptable approximation to the response of the original system?
Consider a structural system that is being studied both experimentally and computationally.
Let $n =$ number of measured dof-s.
Let $N =$ dofs in the computational model.
Typically, $N >> n$.
While reconciling the predictions from the computational model with measured responses in the experimental model, we could either

Reduce the size of the computational model so that only the dof-s which are common to both the experimental and computational models are retained.

Or

Expand the size of the experimental model so that the dof-s in both experimental and computational models match.
Three techniques

• Static condensation (Guyan’s reduction)
• Dynamic condensation
• System equivalent reduction expansion process
Guyan’s reduction technique

\[ X(t) = \begin{cases} X_m(t) \\ X_s(t) \end{cases} = \Psi X_m(t) \]

\[\begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} \ddot{X}_m \\ \ddot{X}_s \end{bmatrix} + \begin{bmatrix} C_{mm} & C_{ms} \\ C_{sm} & C_{ss} \end{bmatrix} \begin{bmatrix} \dot{X}_m \\ \dot{X}_s \end{bmatrix} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} X_m \\ X_s \end{bmatrix} = \begin{bmatrix} F_m(t) \\ 0 \end{bmatrix}\]

Assumption

- Slave dofs carry no nodal forces

Idea: Relate slave dofs to the master dofs through relations which are valid under static conditions. That is use the relation

\[\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} X_m \\ X_s \end{bmatrix} = \begin{bmatrix} F_m(t) \\ 0 \end{bmatrix}\]

to eliminate slave dofs in terms of master dofs.
\[
\begin{bmatrix}
K_{mm} & K_{ms} \\
K_{sm} & K_{ss}
\end{bmatrix}
\begin{bmatrix}
X_m \\
X_s
\end{bmatrix} = \begin{bmatrix}
F_m(t) \\
0
\end{bmatrix}
\]

\[K_{sm}X_m + K_{ss}X_s = 0\]

\[X_s = -K_{ss}^{-1}K_{sm}X_m\]

\[\Rightarrow \text{Transformation matrix}\]

\[\Psi = \begin{bmatrix}
I \\
-K_{ss}^{-1}K_{sm}
\end{bmatrix}\]

\[\begin{bmatrix}
X_m(t) \\
X_s(t)
\end{bmatrix} = \begin{bmatrix}
I \\
-K_{ss}^{-1}K_{sm}
\end{bmatrix}\begin{bmatrix}
X_m(t)
\end{bmatrix}\]

\[T = \frac{1}{2}\dot{X}^{t} M \dot{X} = \frac{1}{2}\dot{X}^{t} \Psi^{t} M \Psi \dot{X} = \frac{1}{2}\dot{X}^{t} M_{r} \dot{X}
\]

\[M_{r} = \Psi^{t} M \Psi\]

\[V = \frac{1}{2}X^{t} K X = \frac{1}{2}X^{t} \Psi^{t} K \Psi X = \frac{1}{2}X^{t} K_{r} X
\]

\[K_{r} = \Psi^{t} K \Psi\]

\[\Rightarrow M_{r} \ddot{X}_m + C_{r} \dot{X}_m + K_{r} X_m = F_r\]
\[ M_r = \Psi^t M \Psi = \begin{bmatrix} I & -K^{-1}_{ss} K_{sm} \\ -K_{ss}^{-1} K_{sm} & M_{mm} & M_{ms} \\ K_{sm} & M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} I & -K_{ss}^{-1} K_{sm} \\ -K_{ss}^{-1} K_{sm} & M_{mm} & M_{ms} \\ K_{sm} & M_{sm} & M_{ss} \end{bmatrix} \]

\[ = M_{mm} - M_{ms} K_{ss}^{-1} K_{sm} - K_{sm}^{-1} K_{ss} K_{sm} M_{sm} + K_{sm}^t K_{ss}^{-1} M_{ss} K_{ss}^{-1} K_{sm} \]

\[ K_r = \Psi^t K \Psi = \begin{bmatrix} I & -K^{-1}_{ss} K_{sm} \\ -K_{ss}^{-1} K_{sm} & K_{mm} & K_{ms} \\ K_{sm} & K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} I & -K_{ss}^{-1} K_{sm} \\ -K_{ss}^{-1} K_{sm} & K_{mm} & K_{ms} \\ K_{sm} & K_{sm} & K_{ss} \end{bmatrix} \]

\[ = K_{mm} - K_{ms} K_{ss}^{-1} K_{sm} - K_{sm}^{-1} K_{ss} K_{sm} + K_{sm}^t K_{ss}^{-1} K_{ss} K_{ss}^{-1} K_{sm} \]

\[ = K_{mm} - K_{ms} K_{ss}^{-1} K_{sm} \]
Remarks

• The slave dofs are related to the master dofs through relations that are strictly valid for static situations and hence this method is also known as method of static condensation.

• The partitioning of dofs as being masters and slaves has to be done by the analyst bearing in mind the following points:
  ○ Slave dofs must contribute little to kinetic energy.
  ○ Select slave dofs such that the lowest eigenvalue of the equation $K_{ss} \alpha = \lambda M_{ss} \alpha$ has the highest value.
  ○ Select slave dofs in regions of high stiffness and low mass.
  ○ Ensure that terms of $M_{ss}$ are small and terms of $K_{ss}$ are large.
  ○ Those dofs which yield the larger values of the ratio $\frac{K_{ii}}{M_{ii}}$ can be selected as slaves.
Remarks (continued)

• The error due to model reduction increases with increase in driving frequencies of interest.
• Any initial conditions specified on slave dofs would not be satisfied.
• The static condensation does not reproduce any of the original natural frequencies of the original analytical model and all the natural frequencies of the reduced models would be higher than those of the full model.
Numerical example

\[ K = 1000 \text{ N/m} \]
\[ M = 10 \text{ kg} \]
\[
K = \begin{bmatrix}
7000 & -3000 & 0 & 0 & 0 & 0 & -2000 \\
-3000 & 7000 & -1000 & -3000 & 0 & 0 & 0 \\
0 & -1000 & 8000 & 0 & 0 & 0 & 0 \\
0 & -3000 & 0 & 5000 & -2000 & 0 & 0 \\
0 & 0 & 0 & -2000 & 3500 & -1500 & 0 \\
-2000 & 0 & 0 & 0 & 0 & -1500 & 3500 \\
\end{bmatrix} \text{ (N/m)}
\]

\[
M = \begin{bmatrix}
10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 15 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 20 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 30 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 20 & 0 \\
\end{bmatrix} \text{ (kg)}
\]

\[
\Phi = \begin{bmatrix}
0.0649 & 0.0703 & 0.0746 & -0.0050 & 0.1493 & 0.2509 \\
0.0774 & 0.0023 & 0.1565 & -0.0107 & 0.1258 & -0.1423 \\
0.0102 & 0.0005 & 0.0369 & -0.2130 & -0.0543 & 0.0143 \\
0.1040 & -0.0671 & 0.1312 & 0.0723 & -0.1622 & 0.0509 \\
0.1273 & -0.0850 & -0.0918 & -0.0187 & 0.0335 & -0.0021 \\
0.1043 & 0.1825 & -0.0439 & 0.0085 & -0.0512 & -0.0345 \\
\end{bmatrix}
\]

\[
\omega = [4.6090 \ 13.0911 \ 13.7122 \ 19.9369 \ 22.7136 \ 29.9605] \text{ rad/s}
\]

\[
K\varphi = \omega^2 M\varphi
\]
Static condensation

\[ K(i,i) \over M(i,i) = \begin{bmatrix} 700.0 & 466.7 & 400.0 & 333.3 & 116.7 & 175.0 \end{bmatrix} \]

Case - 1 Master dofs : 1, 2, and 3; slave dofs : 4, 5, and 6

Eigenvalues associated with \( K_{ss} \alpha = \lambda \alpha \)

\[ \lambda = \begin{bmatrix} 53.89 & 200.00 & 371.11 \end{bmatrix} \text{(rad/s)}^2 \]

\[ \Psi = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1667 & 0.8333 & 0 \\ 0.4167 & 0.5833 & 0 \\ 0.7500 & 0.2500 & 0 \end{bmatrix} \]

\[ \Phi_r = \begin{bmatrix} 0.0944 & 0.0720 & 0.1759 \\ 0.1134 & -0.0416 & -0.1350 \\ 0.0153 & -0.2092 & 0.0774 \end{bmatrix} \]

\[ \omega_r = \{5.5107 \ 19.7498 \ 22.0729\} \text{ rad/s} \]
Static condensation

\[
\frac{K(i,i)}{M(i,i)} = \begin{bmatrix}
700.0 & 466.7 & 400.0 & 333.3 & 116.7 & 175.0 \\
\end{bmatrix}
\]

Case - 2
Master dofs: 4, 5, and 6; slave dofs: 1, 2, and 3

Eigenvalues associated with \( K_{ss} \alpha = \lambda \alpha \)

\[
\lambda = \begin{bmatrix}
290.01 & 419.91 & 856.75 \\
\end{bmatrix} \text{(rad/s)}^2
\]

\[
\Phi_r = \begin{bmatrix}
0.1048 & -0.0785 & -0.1831 \\
0.1286 & -0.0750 & 0.1057 \\
0.1053 & 0.1888 & 0.0058 \\
\end{bmatrix}
\]

\[
\omega_r = \begin{bmatrix}
4.6297 & 13.1467 & 15.1456 \\
\end{bmatrix} \text{rad/s}
\]

Case 2 offers better model
Dynamic condensation technique

\[ M\ddot{X} + KX = F \exp(i\omega t) \]

\[ \Rightarrow \left[ -\omega^2 M + K \right]\{X\} = F \]

\[ D(\omega) = -\omega^2 M + K \quad \text{[Dynamic stiffness matrix]} \]

\[ \Rightarrow DX = F \]

\[ X = \begin{Bmatrix} X_m \\ X_s \end{Bmatrix}_{N \times 1} = \Psi X_m \]

\[ \Rightarrow \]

\[ \begin{bmatrix} D_{mm} & D_{ms} \\ D_{sm} & D_{ss} \end{bmatrix} \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{Bmatrix} F_m \\ 0 \end{Bmatrix} \]

\[ \Rightarrow X_s = -D_{ss}^{-1} D_{sm} X_m \]

\[ \Rightarrow \Psi = \begin{bmatrix} I \\ \left[ -\left[ K_{ss} - \omega^2 M_{ss} \right]^{-1} \left[ K_{sm} - \omega^2 M_{sm} \right] \right] \end{bmatrix} \]
In addition to choosing slave and master dofs, here one also needs to specify the frequency $\omega$ at which the condensation has to be done.

The method requires the determination of inverse of the matrix $[K_{ss} - \omega^2 M_{ss}]$.
Approximation to \( \left[ K_{ss} - \omega^2 M_{ss} \right]^{-1} \)

\[
\begin{align*}
\left[ K_{ss} - \omega^2 M_{ss} \right]^{-1} &= \left( K_{ss} \left[ I - \omega^2 K_{ss}^{-1} M_{ss} \right] \right)^{-1} \\
&= \left[ I - \omega^2 K_{ss}^{-1} M_{ss} \right]^{-1} K_{ss}^{-1} \\
&= \left[ I + \omega^2 K_{ss}^{-1} M_{ss} + \omega^4 K_{ss}^{-1} M_{ss} K_{ss}^{-1} M_{ss} + \ldots \right] K_{ss}^{-1} \\
&\approx \left[ I + \omega^2 K_{ss}^{-1} M_{ss} \right] K_{ss}^{-1}
\end{align*}
\]

\[
X = \left[ I - \left[ I + \omega^2 K_{ss}^{-1} M_{ss} \right] K_{ss}^{-1} \left[ K_{sm} - \omega^2 M_{sm} \right] \right] \{X_m\}
\]

This reduction scheme is called the improved reduction scheme (improvement over the static condensation method).

Avoids the need to invert \( \left[ K_{ss} - \omega^2 M_{ss} \right] \) for every \( \omega \).
Alternatively \[ \left[ K_{ss} - \omega^2 M_{ss} \right]^{-1} \] can also be evaluated in terms of the solution of the eigenvalue problem

\[ K_{ss} \alpha = \lambda M_{ss} \alpha \]

Let \( \Phi \) be the matrix of eigenvectors and \( \Lambda \) be the diagonal matrix of eigenvalues such that

\[ \Phi' M_{ss} \Phi = I \& \Phi' K_{ss} \Phi = \Lambda. \]

\[
\begin{align*}
\left[ K_{ss} - \omega^2 M_{ss} \right] y &= p \\
y &= \Phi u \\
\Rightarrow \left[ K_{ss} - \omega^2 M_{ss} \right] \Phi u &= p \\
\Rightarrow \Phi' \left[ K_{ss} - \omega^2 M_{ss} \right] \Phi u &= \Phi' p \\
\Rightarrow \left[ \Lambda - \omega^2 I \right] u &= \Phi' p \\
\Rightarrow y &= \Phi \left[ \Lambda - \omega^2 I \right]^{-1} \Phi' p \\
\Rightarrow \left[ K_{ss} - \omega^2 M_{ss} \right]^{-1} &= \Phi \left[ \Lambda - \omega^2 I \right]^{-1} \Phi'
\end{align*}
\]

\( \Phi \& \Lambda : \) independent of \( \omega. \)
Numerical example: dynamic condensation

<table>
<thead>
<tr>
<th>$\omega_r$</th>
<th>$\omega = 4.61$</th>
<th>$\omega = 13.09$</th>
<th>$\omega = 13.71$</th>
<th>$\omega = 19.93$</th>
<th>$\omega = 22.71$</th>
<th>$\omega = 29.97$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>4.61</td>
<td>13.09</td>
<td>13.45</td>
<td>19.93</td>
<td>19.79</td>
<td>19.07</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>19.57</td>
<td>13.72</td>
<td>13.71</td>
<td>20.59</td>
<td>22.71</td>
<td>21.35</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>21.40</td>
<td>20.15</td>
<td>20.15</td>
<td>25.45</td>
<td>28.64</td>
<td>29.96</td>
</tr>
</tbody>
</table>

Choice 1
All frequencies in rad/s
Numerical example: dynamic condensation

<table>
<thead>
<tr>
<th>$\omega_r$</th>
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<th>$\omega = 13.09$</th>
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<th>$\omega = 22.71$</th>
<th>$\omega = 29.97$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>4.61</td>
<td>7.16</td>
<td>7.86</td>
<td>10.61</td>
<td>9.93</td>
<td>8.79</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>14.96</td>
<td>13.75</td>
<td>13.71</td>
<td>19.93</td>
<td>22.71</td>
<td>29.97</td>
</tr>
</tbody>
</table>

Choice 2
All frequencies in rad/s

- Choice of $\omega$ matters.
- Those natural frequencies which are close to $\omega$ are predicted well.
In addition to choice of master and slave dofs, the choice of $\omega$ also matters.
Those natural frequencies which are close to $\omega$ are predicted well.
In a harmonic response analysis, $\omega$ can be chosen to be equal to the driving frequency.
If the FRF-s need to be traced over a frequency range, for every value of driving frequency, the condensation needs to be made separately.
Expected to lead to acceptable results if modes are well separated and damping is light.
System equivalent reduction expansion process (SEREP)

- Preserves a collection of normal modes during the reduction process.
- Consider a $N$ dof FE model for a linear vibrating systems.
- Let $\Phi$ denote the $N \times p$ modal matrix that includes the first $p$ modes.
- As before we partition $X$ into master and slave dofs.

This induces the following partition on $\Phi$

$$\Phi = \left\{ \begin{array}{c} \Phi_m \\ \Phi_s \end{array} \right\}$$

Sizes: $\Phi:N \times p$, $\Phi_m : n \times p$, $\Phi_s : (N-n) \times p$

Assume: $n \geq p$.

Introduce a $n \times 1$ vector of generalized coordinates $Z(t)$

$$X(t) = \begin{bmatrix} X_m(t) \\ X_s(t) \end{bmatrix} = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \{Z(t)\}$$
\[ X(t) = \begin{bmatrix} X_m(t) \\ X_s(t) \end{bmatrix} = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \{Z(t)\} \]

\[ \Rightarrow \]

\[ X_m(t) = \Phi_m \{Z(t)\} \]

\[ X_s(t) = \Phi_s \{Z(t)\} \]

\[ \Rightarrow \]

\[ \{Z(t)\} = [\Phi_m]^+ \{X_m(t)\} \text{ where } [\Phi_m]^+ = [\Phi_m^T \Phi_m]^{-1} \Phi_m^T \]

\[ [\Phi_m]^+ = \text{ pseudoinverse of } \Phi_m \]

\[ \Rightarrow \]

\[ X(t) = \begin{bmatrix} X_m(t) \\ X_s(t) \end{bmatrix} = \Psi X_m = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} [\Phi_m^T \Phi_m]^{-1} \Phi_m^T \{X_m(t)\} \]

\[ \Psi = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} [\Phi_m^T \Phi_m]^{-1} \Phi_m^T \]

\[ \Rightarrow M_r = \Psi^T M \Psi \]

\[ K_r = \Psi^T K \Psi \]
Concept of Pseudo-inverse

Pseudo-inverse

Motivation

Consider a linear algebraic equation $x_1 + 5x_2 = 1$. Since we have two variables and only one equation, no unique solution is possible. However if we decide to pick the point that is closest to the origin, then the “solution” is unique.

Consider the matrix equation

$$AX = B$$

where

$$A \rightarrow n \times m$$

$$X \rightarrow m \times 1$$

$$B \rightarrow n \times 1$$
Case 1: $m>n$ (Number of unknowns greater than number of equations)

The solution that minimizes the norm $\|X\|$ is given by

$$X^0 = A^{RM} B$$
$$A^{RM} = A^T \left[ A A^T \right]^{-1}$$

$A^{RM}$ = Right Pseudo-inverse of A.

Case 2: $m<n$ (number of unknowns is less than number of equations)

The “solution” $X^0$ that minimizes the norm $\|AX-B\|$ is given by

$$X^0 = A^{LM} B$$
$$A^{LM} = \left[ A A^T \right]^{-1} A^T$$

$A^{LM}$ = Left Pseudo-inverse of A.
\[
A = \begin{bmatrix}
0.6787 & 0.6555 & 0.2769 & 0.6948 & 0.4387 & 0.1869 \\
0.7577 & 0.1712 & 0.0462 & 0.3171 & 0.3816 & 0.4898 \\
0.7431 & 0.7060 & 0.0971 & 0.9502 & 0.7655 & 0.4456 \\
0.3922 & 0.0318 & 0.8235 & 0.0344 & 0.7952 & 0.6463
\end{bmatrix}
\]

\[
B = \text{pseudo inverse of } A = \begin{bmatrix}
1.2713 & 1.6368 & -1.4333 & -0.3097 \\
1.0286 & -0.6138 & -0.1202 & -0.1329 \\
1.6711 & -0.6109 & -1.2445 & 0.7315 \\
-0.2419 & -0.5423 & 0.9348 & -0.2292 \\
-1.3539 & -0.6393 & 1.4178 & 0.4807 \\
-1.2726 & 0.6308 & 0.6671 & 0.2305
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
1.0000 & -0.0000 & -0.0000 & -0.0000 \\
0.0000 & 1.0000 & -0.0000 & 0.0000 \\
0.0000 & 0.0000 & 1.0000 & -0.0000 \\
-0.0000 & 0.0000 & 0.0000 & 1.0000
\end{bmatrix}
\]

\[ A = \begin{bmatrix}
0.8143 & 0.6160 & 0.9172 & 0.0759 \\
0.2435 & 0.4733 & 0.2858 & 0.0540 \\
0.9293 & 0.3517 & 0.7572 & 0.5308 \\
0.3500 & 0.8308 & 0.7537 & 0.7792 \\
0.1966 & 0.5853 & 0.3804 & 0.9340 \\
0.2511 & 0.5497 & 0.5678 & 0.1299
\end{bmatrix} \]

\[ B = \text{pseudo inverse of } A \]
\[ B = \begin{bmatrix}
0.0722 & 2.4626 & 1.1602 & -1.3753 & 0.5401 & -1.4398 \\
-0.0340 & 3.2902 & -0.6433 & -0.5762 & 0.7234 & -0.4634 \\
0.6329 & -4.1668 & -0.3440 & 1.6690 & -1.2767 & 1.9349 \\
-0.5038 & -0.7280 & 0.4489 & 0.3370 & 0.6602 & -0.3080
\end{bmatrix} \]

\[ AB = \begin{bmatrix}
1.0000 & -0.0000 & -0.0000 & -0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 1.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & 1.0000
\end{bmatrix} \]
TYPICAL RESPONSE ANALYSIS LOOP FOR LINEAR SYSTEMS

Continuum → Discretization → \( M, C, K, F \) → Eigenvalue analysis

\[ M\ddot{X} + C\dot{X} + KX = F(t); X(0), \dot{X}(0) \]

Determine forced response

\[ X(\omega) = H(\omega)F(\omega) \]
\[ X(t) = \int_0^t [h(t-\tau)]F(\tau)d\tau \]

Compute

\[ [H(\omega)] \quad \& \quad [h(t)] \]

MODAL ANALYSIS

\[ K\phi = \omega^2 M\phi \]
\[ \Phi'M\Phi = I; \Phi'K\Phi = \text{diag}[\omega^2] \]
\[ C = \alpha M + \beta K \Rightarrow \Phi'C\Phi \]

\[ \left\{ \omega_i^2 \right\}_{i=1}^n, [\Phi], \left\{ \eta_i \right\}_{i=1}^n, \left\{ \Gamma_i \right\}_{i=1}^n \]
Typical loop for modal testing

Known excitations → Test structure → Response

Experimental modal analysis

- Determine response to Arbitrary loads
- Develop valid FE models
- Detect damage
- Modify structure

Process response to get

\[ [H(\omega)] \quad \& \quad [h(t)] \]

\[ \left\{ \omega_i^2 \right\}_{i=1}^n, [\Phi], \left\{ \eta_i \right\}_{i=1}^n, \left\{ \Gamma_i \right\}_{i=1}^n \]

INVERSE PROBLEM
Numerical example: SEREP

Case - 1 Master dofs: 1, 2, and 3; slave dofs: 4, 5, and 6

Retain the first three modes \((n=3)\).

\[
\Phi_m = \begin{bmatrix}
0.0649 & 0.0703 & 0.0746 \\
0.0774 & 0.0023 & 0.1565 \\
0.0102 & 0.0005 & 0.0369 \\
\end{bmatrix}
\]

\[
\Phi_s = \begin{bmatrix}
0.1040 & -0.0671 & 0.1312 \\
0.1273 & -0.0850 & -0.0918 \\
0.1043 & 0.1825 & -0.0439 \\
\end{bmatrix}
\]

\[
\Psi = \begin{bmatrix}
1.0000 & -0.0000 & 0.0000 \\
-0.0000 & 1.0000 & 0.0000 \\
-0.0000 & -0.0000 & 1.0000 \\
-1.0018 & 3.2893 & -8.3684 \\
-1.2222 & 6.0717 & -25.7591 \\
2.6408 & -0.0157 & -6.4561 \\
\end{bmatrix}
\]
Reduced system natural frequencies:
\[ \omega_r = [4.6090 \ 13.0911 \ 13.7122] \text{ rad/s} \]

Reduced modal matrix
\[ \Phi_r = \begin{bmatrix} 0.8393 & 1.0000 & -0.4765 \\ 1.0000 & 0.0325 & -1.0000 \\ 0.1320 & 0.0071 & -0.2359 \end{bmatrix} \]

Reduced structural matrices
\[ M_r = 10^4 \times \begin{bmatrix} 0.0209 & -0.0273 & 0.0729 \\ -0.0273 & 0.1283 & -0.5103 \\ 0.0729 & -0.5103 & 2.1810 \end{bmatrix} \quad \text{and} \quad K_r = 10^6 \begin{bmatrix} 0.0359 & -0.0464 & 0.1235 \\ -0.0464 & 0.0908 & -0.3313 \\ 0.1235 & -0.3313 & 1.4652 \end{bmatrix} \]
Case - 2 Master dofs : 4, 5, and 6; slave dofs : 1, 2, and 3

Retain the first three modes \((n=3)\).

\[
\Phi_m = \begin{bmatrix}
0.1040 & -0.0671 & -0.1312 \\
0.1273 & -0.0850 & 0.0918 \\
0.1043 & 0.1825 & 0.0439
\end{bmatrix}
\]

\[
\Phi_s = \begin{bmatrix}
0.0649 & 0.0703 & 0.0746 \\
0.0774 & 0.0023 & 0.1565 \\
0.0102 & 0.0005 & 0.0369
\end{bmatrix}
\]

\[
\Psi = \begin{bmatrix}
1.0000 & -0.0000 & 0.0000 \\
-0.0000 & 1.0000 & 0.0000 \\
-0.0000 & -0.0000 & 1.0000 \\
0.5210 & -0.2811 & 0.4462 \\
0.9987 & -0.3758 & 0.2049 \\
0.2107 & -0.1141 & 0.0271
\end{bmatrix}
\]
Reduced system natural frequencies:

\[ \omega_r = \begin{bmatrix} 4.6090 & 13.0911 & 13.7122 \end{bmatrix} \text{ rad/s} \]

Reduced modal matrix

\[ \Phi_r = \begin{bmatrix} 0.5340 & -0.3164 & -0.7901 \\ 0.6539 & -0.4004 & 0.5530 \\ 0.5359 & 0.8600 & 0.2643 \end{bmatrix} \]

Reduced structural matrices

\[ K_r = 10^3 \times \begin{bmatrix} 4.7019 & -3.0944 & -0.2786 \\ -3.0944 & 4.4262 & -1.6702 \\ -0.2786 & -1.6702 & 2.8490 \end{bmatrix} \]

\[ M_r = \begin{bmatrix} 33.5619 & -7.5745 & 5.5088 \\ -7.5745 & 33.1687 & -2.4713 \\ 5.5088 & -2.4713 & 22.6358 \end{bmatrix} \]
The user needs to specify the number of modes to be retained, the mode indices, and the slave and master dofs.

- The choice of normal modes to be included in the reduced model is arbitrary.
- The scheme preserves collection of normal modes during reduction.
- The transformation matrix is deduced from the modal matrix. The modal matrix can be incomplete. Knowledge of $K$ and $M$ is needed. This could be of value if modal matrix is obtained experimentally.
- The natural frequencies of the reduced system matches with the full system natural frequencies irrespective of choice of master and slave dofs.
- The method can be used for model reduction or for model expansion.
Coupling techniques

• Large complex structures require handling of large size matrices.
• Parts of the structure could be modeled experimentally and parts computationally.
• How to develop model for built-up structures based on models for substructures?

A good coupling technique needs to possess the following desirable features:

• Must be versatile enough to accept data either from experiments or from FE models.
• Each component can be treated by an accurate and refined model. Components may have to be broken into small enough subsystems which permit suitable experimental tests/analytical modeling to be carried out.
• Any structural modification which has to be applied at any time only involves a re-analysis of the affected part.
• The technique must permit analysis of different components at different times and by different teams.
Steps

- Partition the physical system into number of substructures with a proper choice of connection and interior coordinates.
- Decide upon the method of analysis for each of the substructures (analytical/experimental)
- Derive the respective subsystem models either by a theoretical or experimental approach.
- Carry out condensation of dofs at the subsystem level. Assess the effect of neglect of certain modes/coordinates.
- Formulate the subsystem equation of motion either using spatial coordinates or modal coordinates. Analysis of one substructure should not require the knowledge of dynamic properties of remaining components.
- Arrive at the reduced order equations for the global structure by invoking interface displacement established for different component models.
Classification

Models for the subsystems

- In terms of structural matrices and spatial coordinates
- In terms of modal parameters (natural frequencies, mode shapes, modal damping ratios, and participation factors)

⇒

1. Impedance coupling techniques
   
   Reduction within the substructure is performed in terms of spatial coordinates or with the help of FRF-s of the subsystems.

2. Modal coupling techniques
   
   Reduced models for the subsystems are obtained in terms of subsystem normal modes