Dynamic Programming Applications

Capacity Expansion

D Nagesh Kumar, IISc

Objectives

- To discuss the Capacity Expansion Problem
- To explain and develop recursive equations for both backward approach and forward approach
- To demonstrate the method using a numerical example

Capacity Expansion Problem

- Consider a municipality planning to increase the capacity of its infrastructure (ex: water treatment plant, water supply system etc) in future
- Sequential increments are to be made in specified time intervals
- The capacity at the beginning of time period *t* be S_t
- Required capacity at the end of that time period be K_t
- Thus, x_t be the added capacity in each time period
- Cost of expansion at each time period can be expressed as a function of S_t and x_t , i.e. $C_t(S_t, x_t)$

Capacity Expansion Problem … contd.

- Optimization problem: To find the time sequence of capacity expansions which minimizes the present value of the total future costs
- ***Objective function:***

  Minimize $\sum_{t=1}^{T} C_t(S_t, x_t)$

- $C_t(S_t, x_t)$: Present value of the cost of adding an additional capacity x_t in the time period *t*
- ***Constraints:*** Capacity demand requirements at each time period
- S_t : Initial capacity

Capacity Expansion Problem … contd.

- Each period's final capacity or next period's initial capacity should be equal to the sum of initial capacity and the added capacity

$$S_{t+1} = S_t + x_t \qquad for\ t = 1,2,...,T$$

- At the end of each time period, the required capacity is fixed

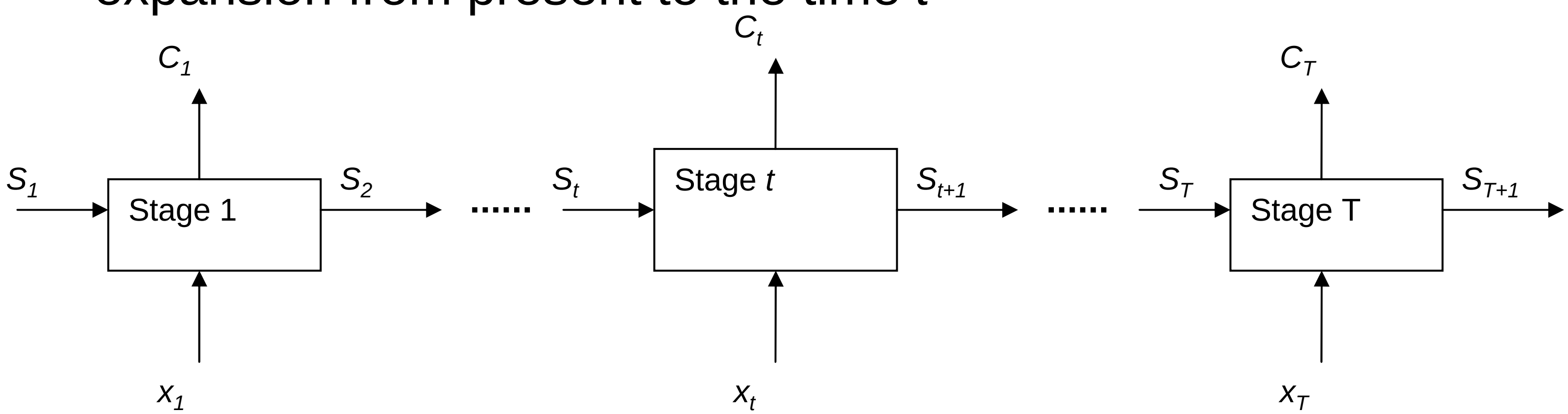
$$S_{t+1} \geq K_t \qquad for\ t = 1,2,...,T$$

- Constraints to the amount of capacity added in each time period i.e. x_t can take only some feasible values.

$$x_t \in \Omega_t$$

Capacity Expansion Problem: Forward Recursion

- Stages of the model: Time periods in which capacity expansion to be made
- State: Capacity at the end of each time period *t*, S_{t+1}
- S_1 : Present capacity before expansion
- $f_t(S_{t+1})$: Minimum present value of total cost of capacity expansion from present to the time *t*



Capacity Expansion Problem: Forward Recursion …contd.

- For the first stage, objective function

$$f_1(S_2) = \min \; C_1(S_1, x_1)$$
$$= \min \; C_1(S_1, S_2 - S_1)$$

- Value of S_2 can be between K_1 and K_T
  - where K_1 is the required capacity at the end of time period 1 and K_T is the final capacity required
- Now, for the first two stages together,

$$f_2(S_3) = \min_{\substack{x_2 \\ x_2 \in \Omega_2}} \left[C_2\left(S_2, x_2\right) + f_1(S_2) \right]$$
$$= \min_{\substack{x_2 \\ x_2 \in \Omega_2}} \left[C_2\left(S_3 - x_2, x_2\right) + f_1(S_3 - x_2) \right]$$

- Value of S_3 can be between K_2 and K_T

Capacity Expansion Problem: Forward Recursion …contd.

- In general, for a time period *t*,

$$f_t(S_{t+1}) = \min_{\substack{x_t \\ x_t \in \Omega_t}} \left[C_t\left(S_{t+1} - x_t, x_t\right) + f_{t-1}(S_{t+1} - x_t) \right]$$

- Subjected to

$$K_t \le S_{t+1} \le K_T$$

- For the last stage, i.e. $t = T$, $f_T(S_{T+1})$ need to be solved only for

$$S_{T+1} = K_T$$

Capacity Expansion Problem: Backward Recursion

- Stages of the model: Time periods in which capacity expansion to be made
- State: Capacity at the beginning of each time period *t*, S_t
- $f_t(S_t)$: Minimum present value of total cost of capacity expansion in periods *t* through *T*
- For the last period *T*, the final capacity should reach K_T after doing the capacity expansions

$$f_T(S_T) = \min_{\substack{x_T \\ x_T \in \Omega_T}} \left[C_T(S_T, x_T) \right]$$

- Value of S_T can be between K_{T-1} and K_T

Capacity Expansion Problem: Backward Recursion …contd.

- In general, for a time period *t*,

$$f_t(S_t) = \min_{\substack{x_t \\ x_t \in \Omega_t}} \left[C_t(S_t, x_t) + f_{t+1}(S_t + x_t) \right]$$

- Solved for all values of S_t ranging from K_{t-1} and K_t
- For period 1, the above equation must be solved only for $S_t = S_1$

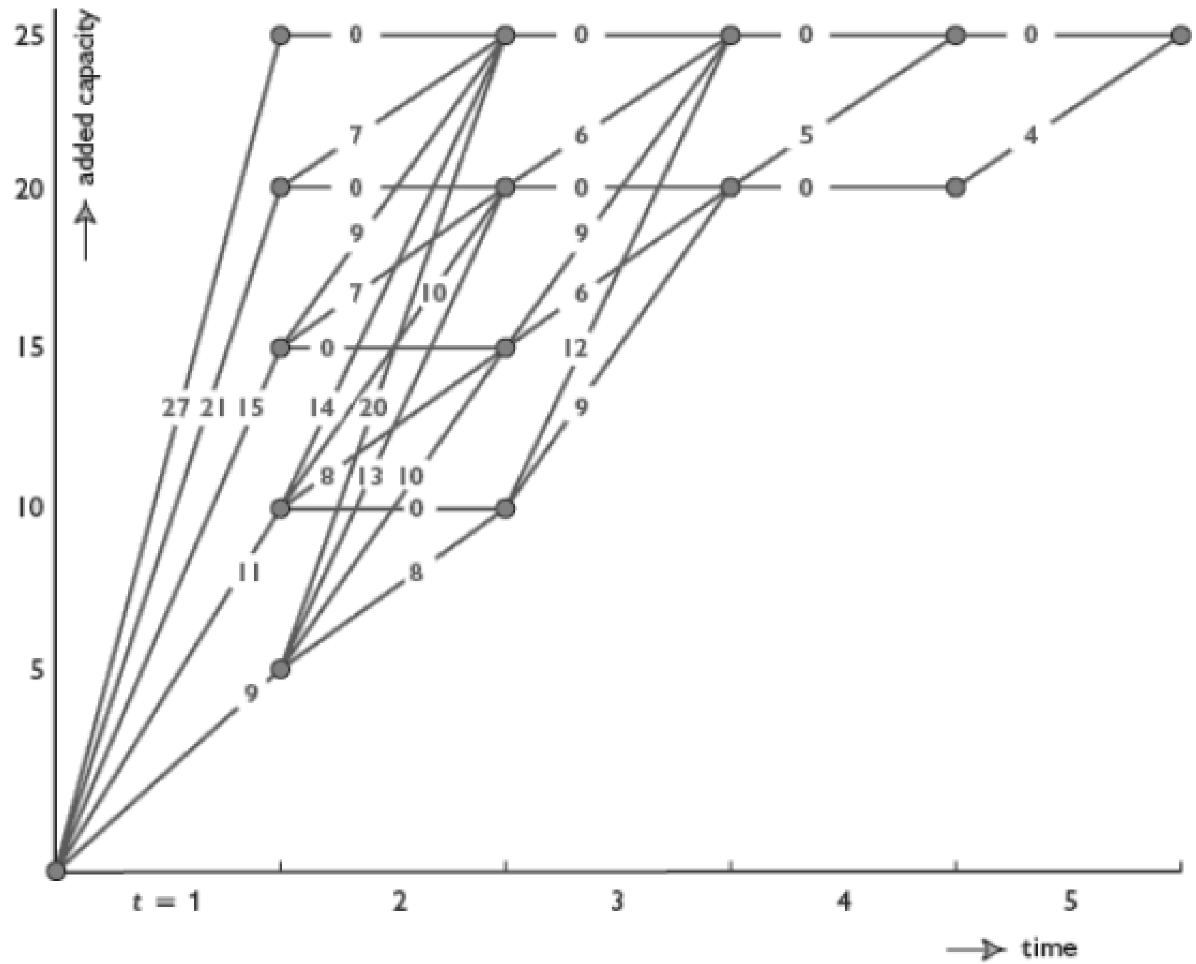
Capacity Expansion: Numerical Example

- Consider a five stage capacity expansion problem
- The minimum capacity to be achieved at the end of each time period is given in the table below

t	D_t
1	5
2	10
3	20
4	20
5	25

Capacity Expansion: Numerical Example …contd.

Expansion costs for each combination of expansion



Numerical Example: Forward Recursion

- Consider the first stage, t =1
- The final capacity for stage 1, S_2 can take values between D_1 to D_5
- Let the state variable can take discrete values of 5, 10, 15, 20 and 25
- Objective function for 1st subproblem with state variable as S_2 can be expressed as

$$f_1(S_2) = \min \; C_1(S_1, x_1)$$
$$= \min \; C_1(S_1, S_2 - S_1)$$

Numerical Example: Forward Recursion …contd.

Computations for stage 1 are given in the table below

Table 2

Stage 1

State Variable, S_2	Added Capacity, $x_1 = S_2 - S_1$	$C_1(S_2)$	$f_1^*(S_2)$
5	5	9	9
10	10	11	11
15	15	15	15
20	20	21	21
25	25	27	27

Numerical Example: Forward Recursion …contd.

- Now considering the 1st and 2nd stages together
- State variable S_3 can take values from D_2 to D_5
- Objective function for 2nd subproblem is

$$f_2(S_3) = \min_{\substack{x_2 \\ x_2 \in \Omega_2}} \left[C_2(S_2, x_2) + f_1(S_2) \right]$$

$$= \min_{\substack{x_2 \\ x_2 \in \Omega_2}} \left[C_2(S_3 - x_2, x_2) + + f_1(S_3 - x_2) \right]$$

- The value of x_2 should be taken in such a way that the minimum capacity at the end of stage 2 should be 10, i.e.

$$S_3 \geq 10$$

Numerical Example: Forward Recursion …contd.

Computations for stage 2 are given in the table below

Table 3

Stage 2

State Variable, S_3	Added Capacity, x_2	$C_2(S_3)$	$S_2 = S_3 - x_2$	$f_1^*(S_2)$	$f_2(S_3)=C_2(S_3)+f_1^*(S_2)$	$f_2^*(S_3)$
10	0	0	10	11	11	11
	5	8	5	9	17	
15	0	0	15	15	15	15
	5	8	10	11	19	
	10	10	5	9	19	
20	0	0	20	21	21	21
	5	7	15	15	22	
	10	10	10	11	21	
	15	13	5	9	22	
25	0	0	25	27	27	24
	5	7	20	21	28	
	10	9	15	15	24	
	15	14	10	11	25	
	20	20	5	9	29	

Numerical Example: Forward Recursion …contd.

- Like this, repeat this steps till t = 5
- The computation tables are shown

Table 4

Stage 3

State Variable, S_4	Added Capacity, x_3	$C_3(S_4)$	$S_3 = S_4 - x_3$	$f_2^*(S_3)$	$f_3(S_4)=C_3(S_4)+f_2^*(S_3)$	$f_3^*(S_4)$
20	0	0	20	21	21	20
	5	6	15	15	21	
	10	9	10	11	20	
25	0	0	25	24	24	23
	5	6	20	21	27	
	10	9	15	15	34	
	15	12	10	11	23	

Numerical Example: Forward Recursion …contd.

Table 5

Stage 4

State Variable, S_5	Added Capacity, x_4	$C_4(S_5)$	$S_4 = S_5 - x_4$	$f_3^*(S_4)$	$f_4(S_5)=C_4(S_5)+f_3^*(S_4)$	$f_4^*(S_5)$
20	0	0	20	20	20	20
25	0	0	25	23	23	23
	5	5	20	20	25	

For the 5th subproblem, state variable $S_6 = D_5$

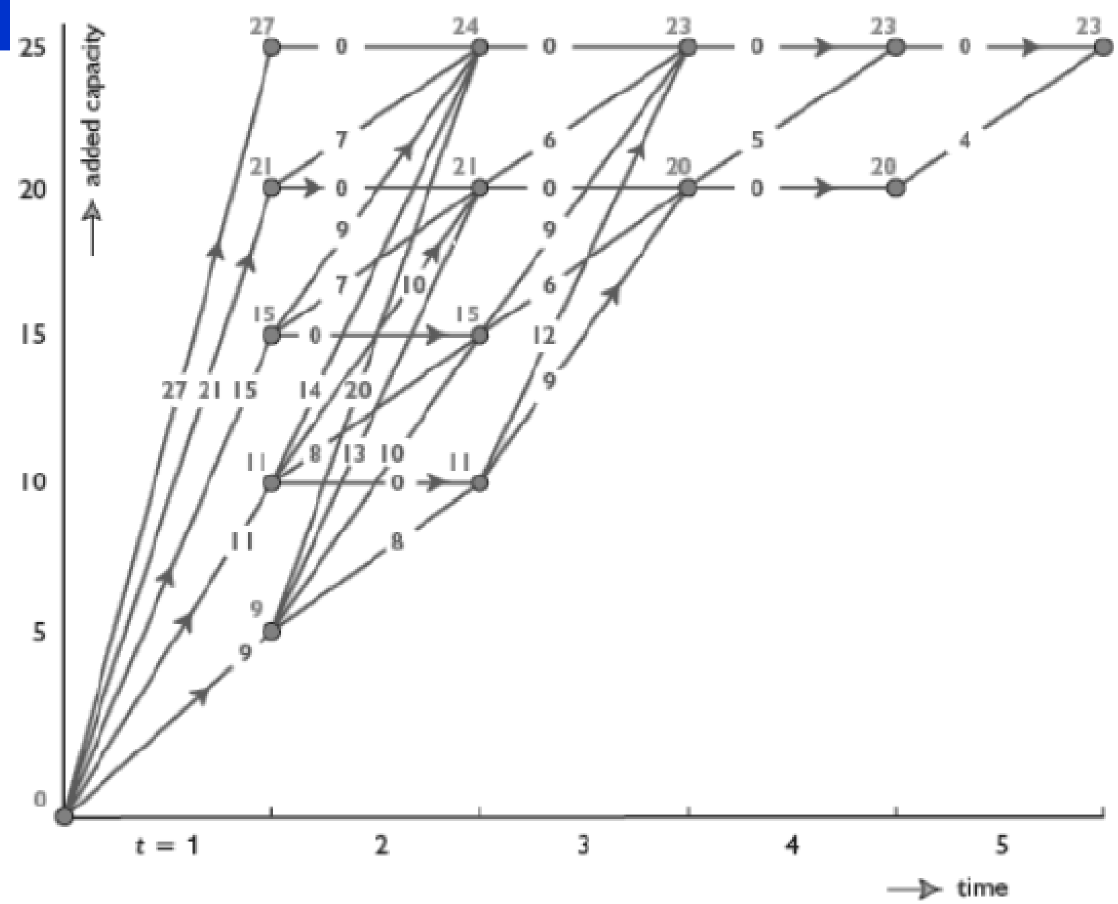
Table 6

Stage 5

State Variable, S_6	Added Capacity, x_5	$C_5(S_6)$	$S_5 = S_6 - x_5$	$f_4^*(S_5)$	$f_5(S_6)=C_5(S_6)+f_4^*(S_5)$	$f_5^*(S_6)$
25	0	0	25	23	23	23
	5	4	20	20	24	

Numerical Example: Forward Recursion …contd.

Figure showing the solutions with the cost of each addition along the links and the minimum total cost at each node



- **Optimal cost of expansion is 23 units**
- **By doing backtracking from the last stage (farthest right node) to the initial stage, the optimal expansion to be done at 1st stage = 10 units, 3rd stage = 15 units and rest all stages = 0 units**

Numerical Example: Backward Recursion

- Capacity at the final stage is given as $S_6 = 25$
- Consider the last stage, t =5
- Initial capacity for stage 5, S_5 can take values between D_4 to D_5
- Objective function for 1st subproblem with state variable as S_5 can be expressed as

$$f_5(S_5) = \min_{\substack{x_T \\ x_T \in \Omega_T}} \left[f_5(S_5, x_5) \right]$$

- The optimal cost of expansion can be achieved by following the same procedure to all stages

Numerical Example: Backward Recursion …contd.

Computations for all stages

Table 7

Stage 5

State Variable, S_5	Added Capacity, x_5	$C_5(S_5)$	$f_5^*(S_5)$
20	5	4	4
25	0	0	0

Table 8

Stage 4

State Variable, S_4	Added Capacity, x_4	$C_4(S_4)$	$S_5 = S_4 + x_4$	$f_5^*(S_5)$	$f_4(S_4)=C_4(S_4)+f_5^*(S_5)$	$f_4^*(S_4)$
20	0	0	20	4	4	4
	5	5	25	0	5	
25	0	0	25	0	0	0

Numerical Example: Backward Recursion …contd.

Table 9

Stage 3

State Variable, S_3	Added Capacity, x_3	$C_3(S_3)$	$S_4 = S_3 + x_3$	$f_4^*(S_4)$	$f_3(S_3)=C_3(S_3)+f_4^*(S_4)$	$f_3^*(S_3)$
10	10	9	20	4	13	12
	15	12	25	0	12	
15	5	6	20	4	10	10
	10	9	25	0	10	
20	0	0	20	4	4	4
	5	6	25	0	5	
25	0	0	25	0	0	0

Numerical Example: Backward Recursion …contd.

Table 10

Stage 2

State Variable, S_2	Added Capacity, x_2	$C_2(S_2)$	$S_3 = S_2 + x_2$	$f_3^*(S_3)$	$f_2(S_2)=C_2(S_2)+f_3^*(S_3)$	$f_2^*(S_2)$
5	5	8	10	12	20	17
	10	10	15	10	20	
	15	13	20	4	17	
	20	20	25	0	20	
10	0	0	10	12	12	12
	5	8	15	10	18	
	10	10	20	4	14	
	15	14	25	0	14	
15	0	0	15	10	10	9
	5	7	20	4	11	
	10	9	25	0	9	
20	0	0	20	4	4	4
	5	7	25	0	7	
25	0	0	25	0	0	0

Numerical Example: Backward Recursion …contd.

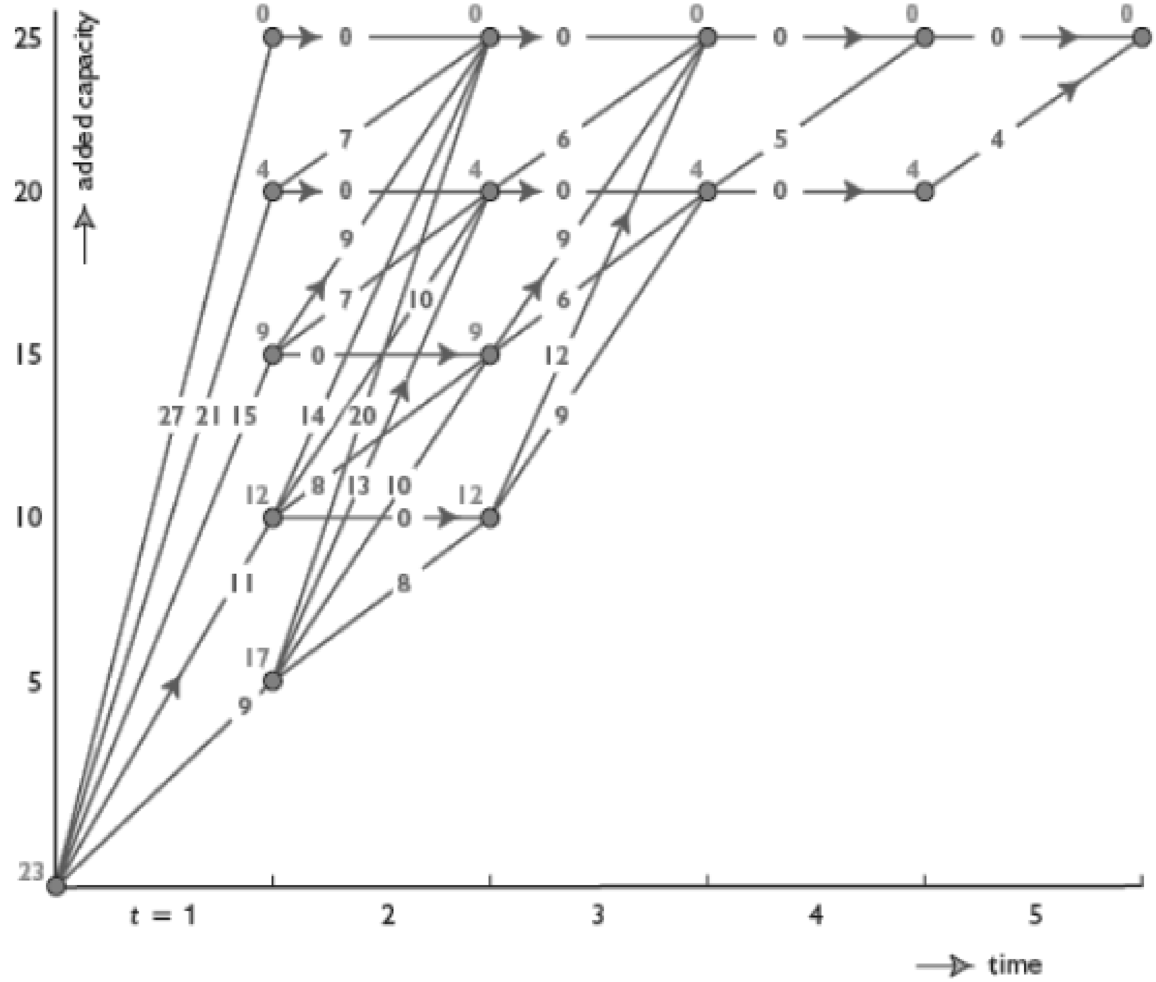
Table 11

Stage 2

State Variable, S_1	Added Capacity, x_1	$C_1(S_1)$	$S_2 = S_1 + x_1$	$f_2^*(S_2)$	$f_1(S_1) = C_1(S_1) + f_2^*(S_2)$	$f_1^*(S_2)$
0	5	9	5	17	26	23
	10	11	10	12	23	
	15	15	15	9	24	
	20	21	20	4	25	
	25	27	25	0	27	

Numerical Example: Backward Recursion …contd.

- Optimal cost of expansion is obtained from the node value at the first node i.e. 23 units
- Optimal expansions to be made are 10 units at the first stage and 15 units at the last stage



Capacity Expansion Problem: Uncertainty

- The future demand and the future cost of expansion in this problem are highly uncertain
- Hence, the solution obtained cannot be used for making expansions till the end period, T
- But, decisions about the expansion to be done in the current period can be very well done through this
- For the uncertainty on current period decisions to be less , the final period T should be selected far away from the current period

Thank You