Linear Programming Applications

Structural & Water Resources Problems

D Nagesh Kumar, IISc

Introduction

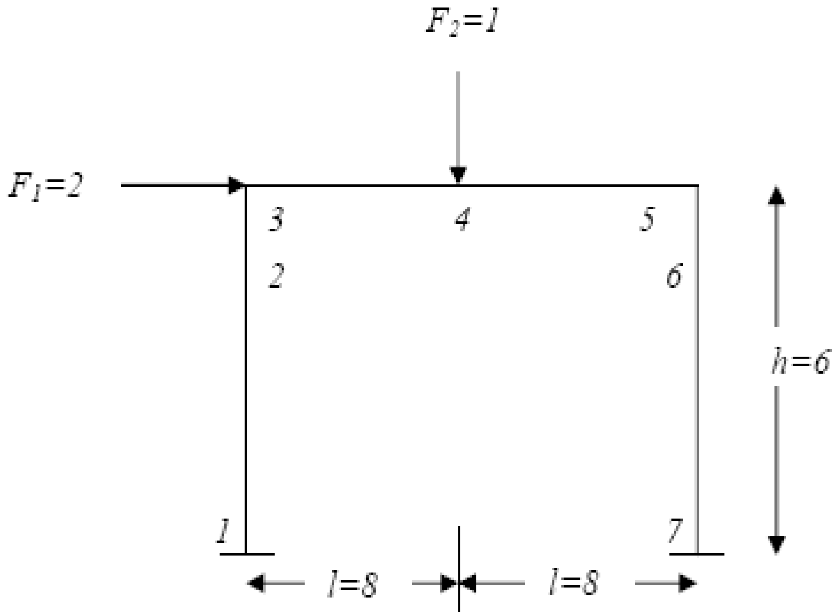
- LP has been applied to formulate and solve several types of problems in engineering field
- LP finds many applications in the field of water resources and structural design which include
  - Planning of urban water distribution
  - Reservoir operation
  - Crop water allocation
  - Minimizing the cost and amount of materials in structural design

Objectives

- To discuss the applications of LP in the plastic design of frame structures
- To discuss the applications of LP in deciding the optimal pattern of irrigation

Example – Structural Design

- A beam column arrangement of a rigid frame is shown
- Moment in beam is represented by M_b
- Moment in column is denoted by M_c.
- $l = 8$ units and $h = 6$ units
- Forces $F_1=2$ units and $F_2=1$ unit.



Assuming that plastic moment capacity of beam and columns are linear functions of their weights; the objective function is to minimize weights of the materials.

Example - Structural Design …contd.

<u>Solution:</u>

- In the limit design, it is assumed that at the points of peak moments, plastic hinges will be developed
- Points of development of peak moments are numbered in the above figure from 1 through 7
- Development of sufficient hinges makes the structure unstable known as a collapse mechanism
- For the design to be safe the energy absorbing capacity of the frame (U) should be greater than the energy imparted by externally applied load (E) for the various collapse mechanisms of the structure

Example - Structural Design …contd.

- The objective function can be written as

  Minimize *f* = *weight of beam* + *weight of column*

$$f = w\left(2lM_b + 2hM_c\right) \qquad (1)$$

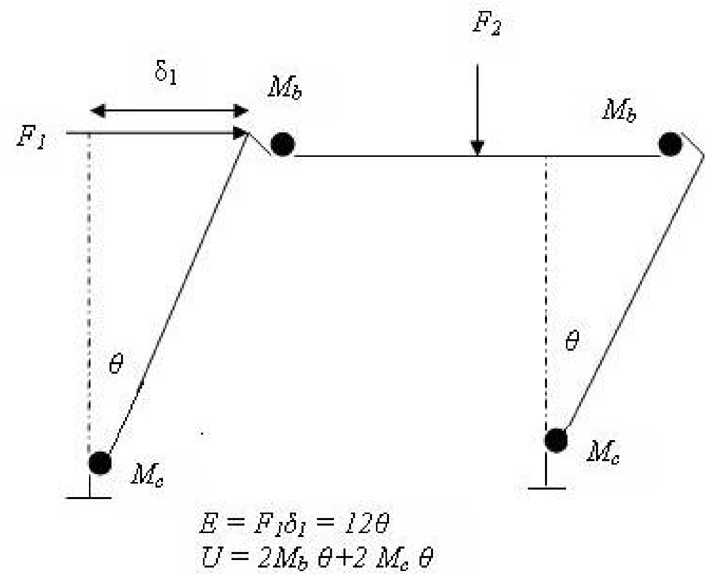
  where *w* is weight per unit length over unit moment in material

- Since *w* is constant, optimizing (1) is same as optimizing

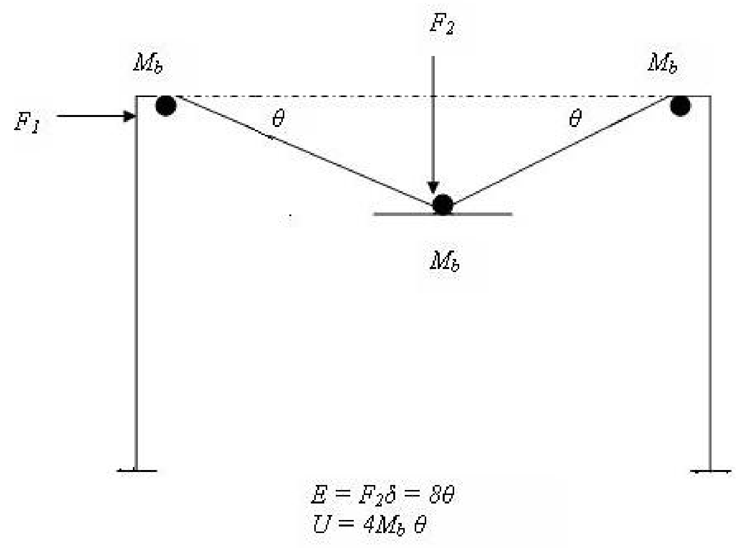
$$\begin{aligned} f &= \left(2lM_b + 2hM_c\right) \\ &= 16M_b + 12M_c \end{aligned}$$

Example - Structural Design …contd.

- Four possible collapse mechanisms are shown in the figure below with the corresponding *U* and *E* values

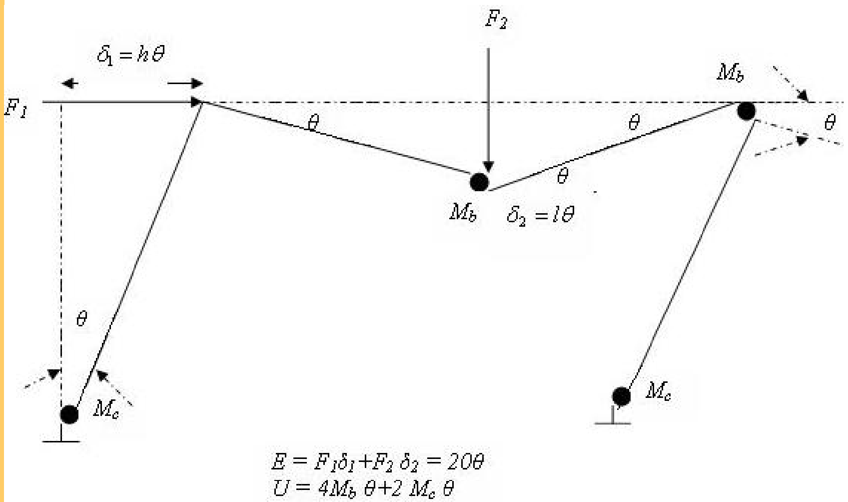


(a)

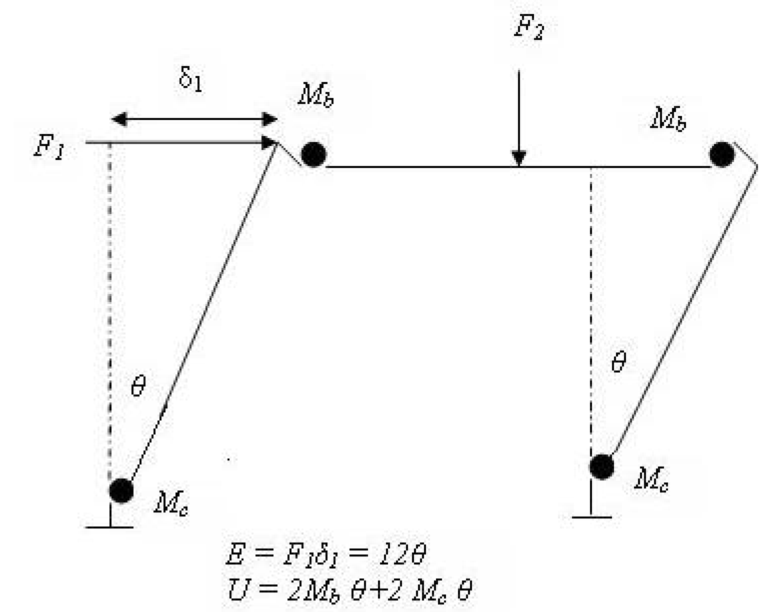


(b)

Example - Structural Design …contd.



(c)



(d)

Example - Structural Design …contd.

- The optimization problem can be stated as

$$Minimize\ \ f = 16M_b + 12M_c$$

subject to

$$M_c \geq 3$$

$$M_b \geq 2$$

$$2M_b + M_c \geq 10$$

$$M_b + M_c \geq 6$$

$$M_b \geq 0; \quad M_c \geq 0$$

Example - Structural Design …contd.

- Introducing slack variables X_1, X_2, X_3, X_4 all , the system of equations can be written in canonical form as

$$16M_B + 12M_C - f = 0$$

$$-M_c + X_1 = -3$$

$$-M_b + X_2 = -2$$

$$-2M_b - M_c + X_3 = -10$$

$$-M_b - M_c + X_4 = -6$$

$$16M_B + 12M_C - f = 0$$

Example - Structural Design …contd.

- This model can be solved using Dual Simplex algorithm
- The final tableau is shown below

The optimal value of decision variables are $M_B = 7/2; M_C = 3$

And the total weight of the material required $f = 92w$ *units*

Iteration 2:

Basic Variables	Variables						b_r
	M_B	M_C	X_1	X_2	X_3	X_4	
f	0	0	-4	0	-8	0	92
M_C	0	1	-1	0	0	0	3
X_2	0	0	½	1	-½	0	3/2
M_B	1	0	½	0	-½	0	7/2
X_4	0	0	-½	0	-½	1	1
Ratio							

Example - Irrigation Allocation

- Consider two crops 1 and 2. One unit of crop 1 produces four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let x be the amount of water required for A units of crop 1 and y be the same for B units of crop 2.
- The amount of production and the amount of water required can be expressed as a linear relation as shown below

$$A = 0.5(x - 2) + 2$$

$$B = 0.6(y - 3) + 3$$

Example - Irrigation Allocation …contd.

- Consider two crops 1 and 2. One unit of crop 1 produces four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let x be the amount of water required for A units of crop 1 and y be the same for B units of crop 2.
- The amount of production and the amount of water required can be expressed as a linear relation as shown below

$$A = 0.5(x - 2) + 2$$

$$B = 0.6(y - 3) + 3$$

Example - Irrigation Allocation …contd.

Solution:

- Objective: Maximize the profit from crop 1 and 2

$$\text{Maximize } f = 4A + 5B;$$

- Expressing as a function of the amount of water,

$$\text{Maximize } \quad f = 4[0.5(x - 2) + 2] + 5[0.6(y - 3) + 3]$$

$$f = 2x + 3y + 10$$

Example - Irrigation Allocation …contd.

subject to

- $x + y \leq 10$; Maximum availability of water
- $x \geq 2$; Minimum amount of water required for crop 1
- $y \geq 3$; Minimum amount of water required for crop 2
- The above problem is same as maximizing

$$f' = 2x + 3y$$

subject to same constraints.

Example - Irrigation Allocation …contd.

- Changing the problem into standard form by introducing slack variables S_1, S_2, S_3

$$Maximize\ f' = 2x + 3y$$

subject to

$$x + y + S_1 = 10$$

$$-x + S_2 = -2$$

$$-y + S_3 = -3$$

This model is solved using simplex method

Example - Irrigation Allocation …contd.

<u>*Iteration 3:*</u>

- The final tableau is as shown

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	0	0	3	1	0	28	-
S_3	0	0	1	1	1	5	-
x	1	0	0	-1	0	2	-
y	0	1	1	1	0	8	-

- The solution is $x = 2; y = 8; \quad f' = 28$

  Therefore, $f = 28+10 = 38$

- Water allocated to crop A is 2 units and to crop B is 8 units and total profit yielded is <u>38 units</u>.

Example – Water Quality Management

- Waste load allocation for water quality management in a river system can be defined as
  - Determination of optimal treatment level of waste, which is discharged to a river
  - By maintaining the water quality standards set by Pollution Control Agency (PCA), through out the river
- Conventional waste load allocation involves minimization of treatment cost subject to the constraint that the water quality standards are not violated

Example - Waster Quality Management …contd.

- Consider a simple problem of *M* dischargers, who discharge waste into the river, and *I* checkpoints, where the water quality is measured by PCA
- Let x_j is the treatment level and a_j is the unit treatment cost for j^{th} discharger $(j=1,2,\ldots,M)$
- c_i is the dissolved oxygen (DO) concentration at checkpoint i $(i=1,2,\ldots,I)$, which is to be controlled
- Decision variables for the waste load allocation model are x_j $(j=1,2,\ldots,M)$.

Example - Waster Quality Management …contd.

- Objective function can be expressed as

$$\text{Maximize} \quad f = \sum_{j=1}^{M} a_j x_j$$

- Relationship between the water quality indicator, c_i (DO) at a checkpoint and the treatment level upstream to that checkpoint is linear (based on Streeter-Phelps Equation)
- Let *g(x)* denotes the linear relationship between c_i and x_j.
- Then, $c_i = g(x_j) \qquad \forall i, j$

Example - Waster Quality Management …contd.

- Let c_P be the permissible DO level set by PCA, which is to be maintained through out the river
- Therefore, $c_i \geq c_P \qquad \forall i$
- This model can be solved using simplex algorithm which will give the optimal fractional removal levels required to maintain the water quality of the river

Thank You