Optimization using Calculus

Kuhn-Tucker Conditions

Introduction

- Optimization with multiple decision variables and equality constraint : Lagrange Multipliers.
- Optimization with multiple decision variables and inequality constraint : Kuhn-Tucker (KT) conditions
- KT condition: Both necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is concave, i.e. the problems belongs to a class called the convex programming problems.

Kuhn Tucker Conditions: Optimization Model

Consider the following optimization problem

Minimize $f(X)$

subject to

$$g_j(X) \leq 0 \qquad for \;\; j=1,2,\ldots,p$$

where the decision variable vector

$$X=[x_1,x_2,\ldots,x_n]$$

Kuhn Tucker Conditions

Kuhn-Tucker conditions for $\mathbf{X}^* = [x_1^* \;\; x_2^* \;\; \ldots \; x_n^*]$ to be a local minimum are

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

$$\lambda_j g_j = 0 \qquad j = 1, 2, \ldots, m$$

$$g_j \leq 0 \qquad j = 1, 2, \ldots, m$$

$$\lambda_j \geq 0 \qquad j = 1, 2, \ldots, m$$

Kuhn Tucker Conditions …contd.

- In case of minimization problems, if the constraints are of the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be non-positive
- On the other hand, if the problem is one of maximization with the constraints in the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be nonnegative.

Example (1)

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$

subject to

$$g_1 = x_1 - x_2 - 2x_3 \leq 12$$

$$g_2 = x_1 + 2x_2 - 3x_3 \leq 8$$

Example (1) …contd.

Kuhn – Tucker Conditions

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$$

$$2x_1 + \lambda_1 + \lambda_2 = 0 \qquad (2)$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \qquad (3)$$

$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \qquad (4)$$

$$\lambda_j g_j = 0$$

$$\lambda_1 (x_1 - x_2 - 2x_3 - 12) = 0 \qquad (5)$$

$$\lambda_2 (x_1 + 2x_2 - 3x_3 - 8) = 0 \qquad (6)$$

$$g_j \leq 0$$

$$x_1 - x_2 - 2x_3 - 12 \leq 0 \qquad (7)$$

$$x_1 + 2x_2 - 3x_3 - 8 \leq 0 \qquad (8)$$

$$\lambda_j \geq 0$$

$$\lambda_1 \geq 0 \qquad (9)$$

$$\lambda_2 \geq 0 \qquad (10)$$

Example (1) …contd.

From (5) either $\lambda_1 = 0$ or $x_1 - x_2 - 2x_3 - 12 = 0$

<u>Case 1</u>

- From *(2), (3) and (4)* we have $x_1 = x_2 = -\lambda_2 / 2$ and $x_3 = \lambda_2 / 2$
- Using these in *(6)* we get $\lambda_2^2 + 8\lambda_2 = 0, \therefore \lambda_2 = 0 \; or \; -8$
- From *(10)*, $\lambda_2 \geq 0$, therefore, $\lambda_2 = 0$,
- Therefore, $\mathbf{X}^* = [\, 0, 0, 0 \,]$

This solution set satisfies all of *(6)* to *(9)*

Example (1) …contd.

<u>Case 2:</u> $x_1 - x_2 - 2x_3 - 12 = 0$

- Using *(2)*, *(3) and (4)*, we have $\frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \frac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0$

  or $17\lambda_1 + 12\lambda_2 = -144$

- But conditions *(9)* and *(10)* give us $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ simultaneously,

  which cannot be possible with $17\lambda_1 + 12\lambda_2 = -144$

  Hence the solution set for this optimization problem is $X^* = [\ 0\ 0\ 0\ ]$

Example (2)

Minimize $f = x_1^2 + x_2^2 + 60x_1$

subject to

$$g_1 = x_1 - 80 \geq 0$$

$$g_2 = x_1 + x_2 - 120 \geq 0$$

Example (2) …contd.

Kuhn – Tucker Conditions

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0 \quad \Rightarrow$$

$$2x_1 + 60 + \lambda_1 + \lambda_2 = 0 \qquad (11)$$

$$2x_2 + \lambda_2 = 0 \qquad (12)$$

$$\lambda_j g_j = 0 \quad \Rightarrow$$

$$\lambda_1 (x_1 - 80) = 0 \qquad (13)$$

$$\lambda_2 (x_1 + x_2 - 120) = 0 \qquad (14)$$

$$g_j \le 0 \quad \Rightarrow$$

$$x_1 - 80 \ge 0 \qquad (15)$$

$$x_1 + x_2 + 120 \ge 0 \qquad (16)$$

$$\lambda_j \ge 0 \quad \Rightarrow$$

$$\lambda_1 \le 0 \qquad (17)$$

$$\lambda_2 \le 0 \qquad (18)$$

Example (2) …contd.

From (13) either $\lambda_1 = 0$ or $(x_1 - 80) = 0$,

<u>Case 1</u>

- From *(11) and (12)* we have $x_1 = -\lambda_2/2 - 30$ and $x_2 = -\lambda_2/2$
- Using these in *(14)* we get $\lambda_2(\lambda_2 - 150) = 0$

  $\therefore \lambda_2 = 0 \; or \; -150$
- Considering $\lambda_2 = 0$, $\mathbf{X}^*$ = [30, 0]. But this solution set violates *(15)* and *(16)*
- For $\lambda_2 = -150$, $\mathbf{X}^*$ = [45, 75]. But this solution set violates *(15)*

Example (2) …contd.

Case 2: $(x_1 - 80) = 0$

- Using $x_1 = 80$ in *(11) and (12)*, we have

$$\lambda_2 = -2x_2$$
$$\lambda_1 = 2x_2 - 220 \qquad (19)$$

- Substitute (19) in (14), we have $-2x_2\left(x_2 - 40\right) = 0$
- For this to be true, either $x_2 = 0 \;\; or \;\; x_2 - 40 = 0$

Example (2) …contd.

- For $x_2 = 0, \; \lambda_1 = -220$
- This solution set violates *(15)* and *(16)*
- For $x_2 - 40 = 0, \; \lambda_1 = -140 \;\; \textit{and} \;\; \lambda_2 = -80$
- This solution set is satisfying all equations from (15) to (19) and hence the desired
- Thus, the solution set for this optimization problem is $\mathbf{X}^*$ = [80 40].

Thank you