



Optimization using Calculus

Kuhn-Tucker Conditions



Introduction

- ❖ Optimization with multiple decision variables and equality constraint : Lagrange Multipliers.
- ❖ Optimization with multiple decision variables and inequality constraint : Kuhn-Tucker (KT) conditions
- ❖ KT condition: Both necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is concave, i.e. the problems belongs to a class called the convex programming problems.



Kuhn Tucker Conditions: Optimization Model

Consider the following optimization problem

$$\text{Minimize } f(X)$$

subject to

$$g_j(X) \leq 0 \quad \text{for } j=1,2,\dots,p$$

where the decision variable vector

$$X = [x_1, x_2, \dots, x_n]$$



Kuhn Tucker Conditions

Kuhn-Tucker conditions for $\mathbf{X}^* = [x_1^* \ x_2^* \ \dots \ x_n^*]$ to be a local minimum are

$$\begin{aligned}\frac{\partial f}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_i} &= 0 & i = 1, 2, \dots, n \\ \lambda_j g_j &= 0 & j = 1, 2, \dots, m \\ g_j &\leq 0 & j = 1, 2, \dots, m \\ \lambda_j &\geq 0 & j = 1, 2, \dots, m\end{aligned}$$



Kuhn Tucker Conditions ...contd.

- ❖ In case of minimization problems, if the constraints are of the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be non-positive
- ❖ On the other hand, if the problem is one of maximization with the constraints in the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be nonnegative.



Example (1)

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$

subject to

$$g_1 = x_1 - x_2 - 2x_3 \leq 12$$

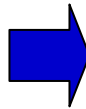
$$g_2 = x_1 + 2x_2 - 3x_3 \leq 8$$



Example (1) ...contd.

Kuhn – Tucker Conditions

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$$

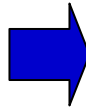


$$2x_1 + \lambda_1 + \lambda_2 = 0 \quad (2)$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \quad (3)$$

$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \quad (4)$$

$$\lambda_j g_j = 0$$



$$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \quad (5)$$

$$\lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \quad (6)$$

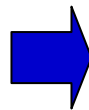
$$g_j \leq 0$$



$$x_1 - x_2 - 2x_3 - 12 \leq 0 \quad (7)$$

$$x_1 + 2x_2 - 3x_3 - 8 \leq 0 \quad (8)$$

$$\lambda_j \geq 0$$



$$\lambda_1 \geq 0 \quad (9)$$

$$\lambda_2 \geq 0 \quad (10)$$



Example (1) ...contd.

From (5) either $\lambda_1 = 0$ or $x_1 - x_2 - 2x_3 - 12 = 0$

Case 1

- From (2), (3) and (4) we have $x_1 = x_2 = -\lambda_2 / 2$ and $x_3 = \lambda_2 / 2$
- Using these in (6) we get $\lambda_2^2 + 8\lambda_2 = 0, \therefore \lambda_2 = 0$ or -8
- From (10), $\lambda_2 \geq 0$, therefore, $\lambda_2 = 0$,
- Therefore, $\mathbf{X}^* = [0, 0, 0]$

This solution set satisfies all of (6) to (9)



Example (1) ...contd.

Case 2: $x_1 - x_2 - 2x_3 - 12 = 0$

➤ Using (2), (3) and (4), we have $\frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \frac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0$

or $17\lambda_1 + 12\lambda_2 = -144$

➤ But conditions (9) and (10) give us $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ simultaneously, which cannot be possible with $17\lambda_1 + 12\lambda_2 = -144$

Hence the solution set for this optimization problem is $X^* = [0 \ 0 \ 0]$



Example (2)

Minimize $f = x_1^2 + x_2^2 + 60x_1$
subject to

$$g_1 = x_1 - 80 \geq 0$$

$$g_2 = x_1 + x_2 - 120 \geq 0$$



Example (2) ...contd.

Kuhn – Tucker Conditions

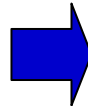
$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$$



$$2x_1 + 60 + \lambda_1 + \lambda_2 = 0 \quad (11)$$

$$2x_2 + \lambda_2 = 0 \quad (12)$$

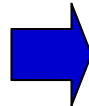
$$\lambda_j g_j = 0$$



$$\lambda_1 (x_1 - 80) = 0 \quad (13)$$

$$\lambda_2 (x_1 + x_2 - 120) = 0 \quad (14)$$

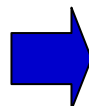
$$g_j \leq 0$$



$$x_1 - 80 \geq 0 \quad (15)$$

$$x_1 + x_2 + 120 \geq 0 \quad (16)$$

$$\lambda_j \geq 0$$



$$\lambda_1 \leq 0 \quad (17)$$

$$\lambda_2 \leq 0 \quad (18)$$



Example (2) ...contd.

From (13) either $\lambda_1 = 0$ or $(x_1 - 80) = 0$,

Case 1

- From (11) and (12) we have $x_1 = -\frac{\lambda_2}{2} - 30$ and $x_2 = -\frac{\lambda_2}{2}$
- Using these in (14) we get $\lambda_2(\lambda_2 - 150) = 0$
 $\therefore \lambda_2 = 0$ or -150
- Considering $\lambda_2 = 0$, $\mathbf{X}^* = [30, 0]$. But this solution set violates (15) and (16)
- For $\lambda_2 = -150$, $\mathbf{X}^* = [45, 75]$. But this solution set violates (15)



Example (2) ...contd.

Case 2: $(x_1 - 80) = 0$

➤ Using $x_1 = 80$ in (11) and (12), we have

$$\lambda_2 = -2x_2$$

$$\lambda_1 = 2x_2 - 220 \quad (19)$$

➤ Substitute (19) in (14), we have $-2x_2(x_2 - 40) = 0$

➤ For this to be true, either $x_2 = 0$ or $x_2 - 40 = 0$



Example (2) ...contd.

- For $x_2 = 0$, $\lambda_1 = -220$
- This solution set violates (15) and (16)
- For $x_2 - 40 = 0$, $\lambda_1 = -140$ and $\lambda_2 = -80$
- This solution set is satisfying all equations from (15) to (19) and hence the desired
- Thus, the solution set for this optimization problem is $\mathbf{X}^* = [80 \ 40]$.



Thank you